ANALYSIS OF A MULTI-DEGREE OF FREEDOM VIBRATING SYSTEM WITH VISCOUS DAMPING USING THE DIGITAL COMPUTER

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Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

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ABSTRACT

The general solution of the behavior of a viscously damped vibrating system having several degrees of freedom, as developed in the literature via matrix methods, is treated in detail here and made the basis of a digital computer program which is capable of determining the natural frequencies, the mode shapes, and the displacements of each mass as functions of time. This program, which is written in FORTRAN 60 language for the Control Data Corporation 1604 computer, affords several output options. It does not treat cases of supercritical damping or cases in which two or more natural frequencies are the same.

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Nomenclature

A	independent variable in reduced system
A'	real part of a complex matrix
B	independent variable in reduced system
B'	imaginary part of a complex matrix
C	damping matrix
D	driving force amplitude
DI	imaginary component of driving force amplitude
DR	real component of driving force amplitude
F(t)	forcing function
G	column of constant coefficients
I	identity matrix
K	stiffness matrix
M	mass or inertia matrix
N	degrees of freedom
O	null matrix
p	eigenvalue
R	independent variable in reduced system
r,s	subscripts designating different modes
SS	steady state solution
TS	transient solution
U	eigenvector
V	real component of eigenvector
W	imaginary component of eigenvector
X	generalized displacement, a column vector
$\dot{X}$	generalized velocity

$\ddot{X}$	generalized acceleration
Y	dependent variable in reduced system
Z	dependent variable in reduced system
α	decrement factor ($\zeta\omega_n$)
β	damped natural frequency
Φ	dependent variable in reduced system
ω	excitation frequency
ω_r	natural frequency of rth mode

CHAPTER I

INTRODUCTION

1.1 General Remarks. The analysis of a subcritically damped multiple degree of freedom vibrating system necessitates obtaining the solution of a complex eigenvalue problem to determine the natural frequencies and mode shapes of the system. Although the analysis presented in the literature for systems with two degrees of freedom may be extended to systems with more than two degrees of freedom, manual calculations are too laborious to be practical. Therefore the natural frequencies are usually found by ignoring the presence of the damping and solving the resulting real eigenvalue problem. This simplifies the problem considerably and provides a good approximation provided the damping is light. Another technique employed is to solve the real eigenvalue problem and then account for the damping using perturbation techniques. However, even in the absence of damping, systems involving more than two degrees of freedom usually require iteration or trial and error techniques (such as the Stodola or Holzer methods) to obtain the mode shapes.

The advent of the electronic digital computer has eliminated the necessity of ignoring the damping component and increased the size of the system for which solutions can be obtained. Although not entirely devoid of error the digital computer is highly reliable and its speed of operation has made it an invaluable tool in engineering analysis and design.

1.2 Method of Analysis. The multiple degree of freedom damped vibrating system is described by a set of N linear second order differential equations, where N denotes the degrees of freedom involved. Utilizing matrix analysis and generalized coordinates the system is described by

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

which is the same form as the single degree of freedom system. However, M, C, and K now represent square matrices and X, $\dot{X}$, $\ddot{X}$, and F(t) are column vectors. The analysis presented in what follows is performed using matrix technique because of the compactness of the notation and the ordered computational procedure which is ideally suited for digital computer programming.

The mathematical analysis is demonstrated and equations are derived which describe the time behavior of free and forced vibrating systems. Finally a digital computer program is presented which performs the operations indicated in the mathematical analysis to determine the natural frequencies, mode shapes, and time behavior of the damped vibrating system.

CHAPTER II

MATHEMATICAL ANALYSIS

2.1 Eigenvalues and Eigenvectors. The viscously damped vibrating system with one degree of freedom is described by a linear, second order differential equation.

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

M, C, and K represent the mass, damping and stiffness of the system, and F(t) the forcing function. A system with many degrees of freedom is described by a set of second order differential equations similar to the case of the single degree of freedom system. Using matrix notation the system is described for the case of free vibration by

$$M\ddot{X} + C\dot{X} + KX = 0 \tag{1}$$

where M, C, K, are now square matrices of order N, representing, as in the single degree case, the mass, damping and stiffness matrices of the system. It is always possible to write the equations so that these matrices are symmetric. N is the number of degrees of freedom. X, $\dot{X}$, and $\ddot{X}$ are column vectors of order N representing displacement, velocity, and acceleration in generalized coordinates.

Premultiply equation (1) by M^{-1} (Capital letters will hereafter represent matrices and lower case letters scalar constants) to obtain;

$$\ddot{X} + M^{-1}C\dot{X} + M^{-1}KX = 0 \tag{1a}$$

W. J. Duncan and A. R. Collar [1]* have shown a simplified method of finding the eigenvalues and eigenvectors by writing the second order differential equation in reduced form as a first order differential equation. The

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* Numbers in brackets refer to references listed in the bibliography page 23

reduced form is obtained by introducing velocity as a dependent variable.

Let

(2a) $$Y = \begin{bmatrix} X \\ \dot{X} \end{bmatrix}$$

Here Y is a partioned column matrix of order 2N with the N displacement components in the upper half and N velocity components in the lower half.

Let

(2b) $$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

Here 0 is an nth-order null matrix and I is an nth-order identity matrix. The reduced equation then becomes

(3) $$\dot{Y} = AY$$

A solution of a first order differential equation such as equation (3) may be taken as

(4) $$Y = u e^{pt}$$

Substitution of the assumed solution into equation (3) gives

(5) $$p e^{pt} u = A e^{pt} u$$

Dividing out the exponential term and rearranging, we have

(5a) $$[A - pI]\, u = 0$$

To have a nontrivial solution, it then follows that

$$\det [A - pI] = 0$$

This is the characteristic matrix of the reduced system. The eigenvalues or roots of the characteristic equation are p_r; r = 1, 2, ...2N and the eigenvectors are U_r; r = 1, 2, ...2N

where

$$u_r = \begin{bmatrix} u_{1,r} \\ u_{2,r} \\ \cdot \\ \cdot \\ u_{2N,r} \end{bmatrix}$$

The first subscript indicates the position in the column and the second subscript corresponds to the r^{th} eigenvalue.

As long as the damping is less than critical in each mode, the reduced system will yield N complex conjugate pairs of eigenvalues and eigenvectors. The eigenvalues will be of the same form as the sub-critically damped single degree of freedom system.

$$p_r = -\zeta_r \omega_r + j\omega_r\sqrt{1-\zeta_r^2}$$

$$\bar{p}_r = -\zeta_r \omega_r - j\omega_r\sqrt{1-\zeta_r^2}$$

The bar indicates the complex conjugate and the subscript indicates the mode, and where

ζ_r - damping ratio

ω_r - natural frequency

$\omega_r\sqrt{1-\zeta_r^2}$ - damped natural frequency

$\zeta_r\omega_r$ - decrement factor

The corresponding eigenvectors are

$$u_r = V_r + jW_r$$

$$\bar{u}_r = V_r - jW_r$$

2.2 Homogeneous Equation.

2.2.1 Orthogonality Relations.

In the case of the undamped vibrating system it is always possible to choose a set of coordinates in which the mass and stiffness matrices are uncoupled. However in the damped vibrating system, unless the damping matrix is proportional to either the mass or stiffness matrix, a set of coordinates which will uncouple the equations

of motion cannot be found without a knowledge of the eigenvalues and eigenvectors of the system.

K. A. Foss has developed a set of orthogonality relations for the eigenvectors of a damped linear dynamic system from which coordinates which uncouple the equations of motion may be found. [2] The technique of reducing the second order differential equation to one of the first order is again employed, with slightly different notation.

Let

(6a) $$R = \begin{bmatrix} 0 & M \\ M & C \end{bmatrix}$$ (6b) $$B = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}$$ (6c) $$Z = \begin{bmatrix} \dot{X} \\ X \end{bmatrix}$$

Using the above notation, equation (1) becomes

(7) $$R\dot{Z} + BZ = 0$$

Furthermore let

$$X = U_r e^{p_r t}$$

$$\dot{X} = p_r U_r e^{p_r t}$$

Then

(8) $$Z = \Phi_r e^{p_r t}$$

where $$\Phi_r = \begin{bmatrix} p_r U_r \\ U_r \end{bmatrix}$$

Substituting equation (8) into equation (7) and dividing out the exponential factor we get,

(9) $$p_r R \Phi_r + B\Phi_r = 0$$

Since the eigenvalues and eigenvectors occur in complex conjugate pairs the complex conjugates of U_r and p_r also satisfy the homogeneous equation.

Denoting the complex conjugate by the subscript S, equation (9) may be written

(10) $$p_s R \Phi_s + B \Phi_s = 0$$

Or since M, C, and K are symmetric, R and B must be symmetric, and equation (10) may be written

(10a) $$p_s \Phi_s^T R + \Phi_s^T B = 0$$

Premultiply equation (9) by Φ_s^T , postmultiply equation (10a) by Φ_r and subtract (10a) from (9).

$$\begin{array}{r}
p_r \Phi_s^T R \Phi_r + \Phi_s^T B \Phi_r = 0 \\
- \; p_s \Phi_s^T R \Phi_r + \Phi_s^T B \Phi_r = 0 \\
\hline
(p_r - p_s) \Phi_s^T R \Phi_r \qquad\qquad = 0
\end{array}$$

Therefore unless r = s the orthogonality relations are

(11) $$\Phi_s^T R \Phi_r = 0$$

(12) $$\Phi_s^T B \Phi_r = 0$$

2.2.2. <u>General solution of the homogeneous equation</u>. In the general solution of equation (7), 2N arbitrary constants of integration must be evaluated. The 2N initial conditions of displacement and velocity are used to evaluate these constants. Assume a solution of the form

(13) $$z(t) = \sum_{r=1}^{2N} \Phi_r C_r e^{pt}$$

where C_r represents the 2N constants to be determined.

The initial conditions may be expressed as a vector expansion of the natural modes

$$(14) \qquad Z(0) = \sum_{r=1}^{2N} \Phi_r C_r$$

Premultiply both sides of equation (14) by $\Phi_s^T R$

$$(15) \qquad \Phi_s^T R Z(0) = \sum_{r=1}^{2N} \Phi_s^T R \Phi_r C_r$$

Using the orthogonality relation, equation (11), the summation simplifies and equation (15) becomes

$$(16) \qquad \Phi_s^T R Z(0) = \Phi_s^T R \Phi_s C_s$$

Or

$$(17) \qquad C_s = \frac{\Phi_s^T R Z(0)}{\Phi_s^T R \Phi_s}$$

The general solution is therefore found by substituting the value of the constants of integration into equation (13).

$$(18) \qquad Z(t) = \sum_{r=1}^{2N} \frac{\Phi_r \Phi_r^T R}{\Phi_r^T R \Phi_r} Z(0)\, e^{pt}$$

It follows from an evaluation of equation (18) at time zero, that

$$\sum_{r=1}^{2N} \frac{\Phi_r \Phi_r^T R}{\Phi_r^T R \Phi_r} = I$$

Equation (18) may be written in the following partitioned form using the nth-order system notation.

$$(19) \qquad \begin{bmatrix} \dot{X}(t) \\ X(t) \end{bmatrix} = \sum_{r=1}^{2N} \frac{\begin{bmatrix} p_r u_r u_r^T M \dot{X}(0) + p_r^2 u_r u_r^T M X(0) + p_r u_r u_r^T C X(0) \\ u_r u_r^T M \dot{X}(0) + p_r u_r u_r^T M X(0) + u_r u_r^T C X(0) \end{bmatrix}}{2 p_r u_r^T M u_r + u_r^T C u_r} e^{p_r t}$$

from which

$$(20) \qquad X(t) = \sum_{r=1}^{2N} \frac{\left[u_r u_r^T M \dot{X}(0) + p_r u_r u_r^T M X(0) + u_r u_r^T C X(0) \right]}{2 p_r u_r^T M u_r + u_r^T C u_r} e^{p_r t}$$

It should be noted again at this point that for the sub-critically damped system being considered, p_r and U_r are complex quantities and occur in complex conjugate pairs. The matrix operations indicated in equation (20) have been carried out by breaking up the complex quantities into real and imaginary parts and then recombining. The operations are relatively straightforward but lengthy, therefore only the final form will be presented.

$$(21) \quad X(t) = \sum_{r=1}^{N} \frac{2}{a_r^2 + b_r^2} \left\{ \left[G_r M \dot{X}(0) + \left(-\alpha_r G_r M + \beta_r H_r M + G_r C \right) X(0) \right] \cos \beta_r t \right\} e^{-\alpha_r t}$$

$$+ \sum_{r=1}^{N} \frac{2}{a_r^2 + b_r^2} \left\{ \left[H_r M \dot{X}(0) + \left(-\alpha_r H_r M - \beta_r G_r M + H_r C \right) X(0) \right] \sin \beta_r t \right\} e^{-\alpha_r t}$$

Where:

$$p_r = -\alpha_r + j\beta_r \qquad u_r = V_r + jW_r$$

$$a_r = -2\alpha_r \left(V_r^T M V_r - W_r^T M W_r \right) - 4\beta_r V_r^T M W_r + V_r^T C V_r - W_r^T C W_r$$

$$b_r = 2\beta_r \left(V_r^T M V_r - W_r^T M W_r \right) - 4\alpha_r V_r^T M W_r + 2 V_r^T C W_r$$

$$A_r = V_r V_r^T - W_r W_r^T$$

$$B_r = V_r W_r^T + W_r V_r^T$$

$$G_r = a_r A_r + b_r B_r$$

$$H_r = b_r A_r - a_r B_r$$

The factor of two in the above equation results from the reduction of the summation from 2N to N. When the indicated operations of equation (20) are carried out from r = 1 to r = 2N the imaginary components sum to zero; therefore all quantities in equation (21) are real. Equation (21) has been given by S. H. Crandall and R. B. McCalley in reference 3.

2.3 Non-Homogeneous Equation

2.3.1 Steady State Solution of the Non-Homogeneous Equation. Once again using the reduced form, the non-homogeneous equation becomes

(22) $$R\dot{Z} + BZ = F(t)$$

Where

$$F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

In a linear nth-order differential equation the solution of the non-homogeneous equation may be found, once the solution of the homogeneous equation is known, by expanding Z in a modal series. [2,4]

Thus,

(23) $$Z(t) = \sum_{r=1}^{2N} \Phi_r Y_r(t)$$

here $Y_r(t)$ is a variable parameter.

Upon substitution of equation (23) in equation (22), we obtain

(24) $$\sum_{r=1}^{2N} R\Phi_r \dot{Y}_r + B\Phi_r Y_r = F(t)$$

Premultiplication of equation (24) by Φ_r^T and making use of the orthogonality relations, yields

(25) $$\Phi_r^T R \Phi_r \dot{Y}_r + \Phi_r^T B \Phi_r Y_r = \Phi_r^T F(t)$$

Or

(25a) $$R_n \dot{Y} - p_r R_n Y = F_n$$

Where

$$R_n = \Phi_r^T R \Phi_r$$

$$-p_r R_n = \Phi_r^T B \Phi_r$$

$$F_n = \Phi_r^T F(t)$$

Using the Laplace Transform technique, the solution of equation (25a) is easily determined.

Rearranging

(a) $\dot{Y} - p_r Y = \frac{F_n}{R_n}$

The Laplace Transform is

(b) $sY(s) - p_r Y(s) = \frac{F_n(s)}{R_n}$

Therefore

(c) $Y(s) = \frac{1}{R_n} \cdot \frac{F_n(s)}{(s-p_r)}$

The inverse transform of (c) is

(26) $$Y(t) = \frac{1}{R_n}\int_0^t e^{p_r(t-\tau)} F_n(\tau)\, d\tau$$

Since $p_r = -\alpha_r + j\beta_r$

(26a) $$Y(t) = \frac{1}{R_n}\int_0^t e^{-\alpha_r(t-\tau)} F_n(\tau)\left(\cos\beta_r(t-\tau) + j\sin\beta_r(t-\tau)\right) d\tau$$

Substitute (26a) into (23)

(27) $$Z(t) = \sum_{r=1}^{2N} \frac{\Phi_r \Phi_r^T}{R_n}\int_0^t e^{-\alpha_r(t-\tau)} F(\tau)\left(\cos\beta_r(t-\tau) + j\sin\beta_r(t-\tau)\right) d\tau$$

As in the case of the homogeneous solution the reduced system quantities are replaced by their nth-order equivalence. The final form of the steady state solution then becomes

(28) $$X(t) = \sum_{r=1}^{N} \frac{2G_r}{a_r^2+b_r^2}\int_0^t f(\tau)\, e^{-\alpha_r(t-\tau)} \cos\beta_r(t-\tau)\, d\tau$$
$$+ \sum_{r=1}^{N} \frac{2H_r}{a_r^2+b_r^2}\int_0^t f(\tau)\, e^{-\alpha_r(t-\tau)} \sin\beta_r(t-\tau)\, d\tau$$

The notation in equation (28) is the same as that used in the homogeneous solution.

2.3.2 Steady State Solution with Sinusoidal Excitation. A special, but very common problem is the case in which the forcing function, $f(t)$, is sinusoidal. Expressing the forcing function as

$$f(t) = \mathcal{R}\left(De^{j\omega t}\right) \tag{29}$$

where $\mathcal{R}$ - indicates "the real part of"

D - column of driving force amplitudes, possibly complex to admit phasing

ω - excitation frequency

The non-homogeneous equation therefore becomes

$$M\ddot{X} + C\dot{X} + KX = \mathcal{R}\left(De^{j\omega t}\right) \tag{30}$$

Assume a solution of the form

$$X(t) = \mathcal{R}\left(Ge^{j\omega t}\right) \tag{31}$$

where G is a column of undetermined constant coefficients. Substitute the assumed solution in equation (30) and divide through by the exponential factor to obtain

$$-\omega^2 MG + j\omega CG + KG = D \tag{32}$$

Or

$$G = \left[K - \omega^2 M + j\omega C\right]^{-1} D \tag{33}$$

Therefore the steady state solution for the case of sinusoidal excitation is

$$X(t) = \mathcal{R}\left\{\left[K - \omega^2 M + j\omega C\right]^{-1} De^{j\omega t}\right\} \tag{34}$$

2.4 Transient Plus Steady State Solution.

2.4.1. Initial Conditions.

The total solution is given by

$$(35)\quad X(t) = TS + SS$$

where TS is the so called transient solution and SS the so called steady state solution. However in the total solution the initial conditions of displacement and velocity in the transient solution must account for the initial values in the steady state solution. This is accomplished as follows. In the case of sinusoidal excitation the steady state solution was found to be

$$(34)\quad X(t) = \mathcal{R}\left\{\left[K - \omega^2 M + j\omega C\right]^{-1} D e^{j\omega t}\right\}$$

To allow for phase differences in the exciting forces on the masses, let D be given by

$$(36)\quad D = DR + j\,DI$$

and let the inverse of $\left[K - \omega^2 M + j\omega C\right]$ be denoted by

$$(37)\quad A' + jB'$$

Then since $e^{j\omega t} = \cos\omega t + j\sin\omega t$ equation (34) becomes

$$(38)\quad X(t) = \mathcal{R}\left\{\left[A' + jB'\right]\left\{DR + jDI\right\}\left(\cos\omega t + j\sin\omega t\right)\right\}$$

Carrying out the indicated multiplications, we obtain

$$(39)\quad X(t) = \left\{(A'DR - B'DI)\cos\omega t - (B'DR + A'DI)\sin\omega t\right\}$$

and the velocity is given by

(40) $\dot{X}(t) = \left\{ -\omega\left(A'DR - B'DI\right)\sin\omega t - \omega\left(B'DR + A'DI\right)\cos\omega t \right\}$

Therefore the initial values of displacement and velocity of the steady state solution are obtained by evaluating equations (38) and (39) at $t = 0$

(41) $X_{SS}(0) = A'DR - B'DI$

(42) $\dot{X}_{SS}(0) = -\omega\left(B'DR + A'DI\right)$

The initial conditions used in determining the time behavior of the transient portion for the total solution therefore become

(43) $X_c(0) = X(0) - \left(A'DR - B'DI\right)$

(44) $V_c(0) = \dot{X}(0) + \omega\left(B'DR + A'DI\right)$

The complete time-solution is thus obtained as indicated in equation (35) where the transient solution is found by using the modified initial conditions of equations (43) and (44).

CHAPTER III

PROGRAM DESCRIPTION

3.1 General Remarks. A digital computer program "PROGRAM VIBSYS" is presented which performs the mathematical operations of the equations developed in the previous chapter. The program is coded in Fortran 60 programming language, [5,6] specifically for the Control Data Corporation 1604 computer. Although the Fortran 60 language is applicable to most large digital computers there are minor variations which are peculiar to the specific system in use. The Fortran 60 language does not permit automatic operations with complex numbers; therefore all operations involving complex numbers are accomplished by operating on the real and imaginary parts separately and then recombining.

The eigenvalues and eigenvectors of the reduced system are found by a matrix iteration scheme which utilizes the direct and inverse power methods and matrix deflation. A mathematical subroutine MATSUB is used to carry out these operations. [7,8] As presented, a maximum of twenty iterations will be performed using the direct power method and then a maximum of twenty using the inverse power method. If convergence is not reached with the inverse power method a print out to this effect will be executed.

Subroutine "INVERT" is used for matrix inversion.* "INVERT" uses the Gaussian elimination and pivotal techniques. Inversion of the $\left[K - \omega^2 M + j\omega C\right]$ matrix which contains complex elements is achieved in

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* Subroutine INVERT is a library subroutine of the Naval Postgraduate School and is designated locally as F1 NPS INVERT.

the following manner. Let $[A + jB]$ be the matrix to be inverted and $[C + jD]$ the inverse to be determined. Then by definition

$$[A + jB][C + jD] = I$$

The above matrix equation leads to two simultaneous equations with two unknowns, C and D.

$$\text{(a)} \quad AC - BD = I$$

$$\text{(b)} \quad BC + AD = 0$$

Solving first for C

$$C = \left[A + BA^{-1}B\right]^{-1}$$

and then for D

$$D = -A^{-1}BC$$

Thus the complex inversion is reduced to multiplications, additions, and inversions, of matrices of real numbers.

Flexibility and utility are the principal aims of the program. Usage requires a knowledge of the mass, stiffness, and damping matrices. Although various output options are available which require additional input data, the three above mentioned matrices are all that are necessary to determine the eigenvalues and eigenvectors of the reduced system and the natural frequencies and mode shapes of the original system.

3.2 Program Options. In addition to finding the natural frequencies and mode shapes of the system, five options are available which describe the time behavior of the system under conditions of free and forced vibration.

(a) Option 1. The time solution of the free vibration problem is obtained in general form and no additional input data is required for execution of this option. In section 2.2.2 the general solution of the homogeneous equation was developed and is repeated here for convenience.

$$(21)\quad X(t) = \sum_{r=1}^{N} \frac{2}{a_r^2 + b_r^2}\left\{\left[G_r M \dot{X}(0) + \left(-\alpha_r G_r M + \beta_r H_r M + G_r C\right) X(0)\right] e^{-\alpha_r t} \cos \beta_r t\right.$$

$$+ \sum_{r=1}^{N} \frac{2}{a_r^2 + b_r^2}\left\{\left[H_r M \dot{X}(0) + \left(-\alpha_r H_r M - \beta_r G_r M + H_r C\right) X(0)\right] e^{-\alpha_r t} \sin \beta_r t\right.$$

The output of option 1 consists of the four coefficient matrices of the $\dot{X}(0)\cos\beta_r t$, $X(0)\cos\beta_r t$, $\dot{X}(0)\sin\beta_r t$ and $X(0)\sin\beta_r t$ terms. Therefore, the output of option 1 will consist of 4N square matrices.

(b) Option 2. The execution of option 2 requires as additional input data the values of the initial displacement, and initial velocity vectors. The product of the coefficient matrices of option 1 and the initial displacement vector or initial velocity vector, as appropriate, is performed to obtain a coefficient column for the cosine and sine terms.

(c) Option 4.* The general steady state solution of the forced vibration problem is provided by option 4. The general solution with a forcing function f(t) was developed in section 2.31 and found to be

$$(28)\quad X(t) = \sum_{r=1}^{N} \frac{2G_r}{a_r^2 + b_r^2} \int_0^t f(\tau)\, e^{-\alpha_r (t-\tau)} \cos \beta_r (t-\tau)\, d\tau$$

$$+ \sum_{r=1}^{N} \frac{2H_r}{a_r^2 + b_r^2} \int_0^t f(\tau)\, e^{-\alpha_r (t-\tau)} \sin \beta_r (t-\tau)\, d\tau$$

In this case the coefficient matrices of the convolution integral are evaluated. There will be 2N square matrices, N corresponding to the coefficient of the integral involving the cosine term and N coefficient matrices of the integral involving the sine term.

(d) Option 5. The steady state solution of the special case of forced vibration with sinusoidal excitation is determined in option 5.

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* Option 3 is described in subsection (e), following.

In section 2.4.1 the expression for the time behavior was found to be

$$X(t) = (A'DR - B'DI) \cos \omega t - (B'DR + A'DI) \sin \omega t \tag{39}$$

Additional input data required for the execution of this option include the driving force amplitude and excitation frequency. The output consists of the coefficient column vectors of the cosine and sine terms.

(e) Option 3. A plot of displacement versus time is made for each mass, for one of the three following cases.

1. Free Vibration
2. Steady state vibration with sinusoidal excitation
3. Transient plus steady state with sinusoidal excitation

Case 1 is obtained when the initial conditions of displacement and velocity are given as input data and the driving force amplitudes and excitation frequency are zero.

Case 2 is obtained when the driving force amplitudes and excitation frequency are given and the initial conditions of displacement and velocity are zero.

Case 3 is obtained when the initial conditions, driving force amplitudes and excitation frequency are all given.

Any combination of the options may be obtained with one set of input data. Input data format and output control is described in detail in Appendix B.

3.3 <u>Accuracy of Method</u>. The accuracy of the solution is contingent on the accuracy with which the eigenvalues and eigenvectors are determined and the effect of computer roundoff error. Internal checks have been provided so that the integrity of the solution may readily be evaluated.

3.3.1 Internal Checks. Prior to the determination of the eigenvalues and eigenvectors the trace and determinate of the matrix, A, is calculated. A check is therefore readily available since the trace of the matrix A is equal to the sum of the eigenvalues of the characteristic matrix of A and the determinant is equal to the product of the eigenvalues. The trace and determinant of A and the sum and product of the eigenvalues are included as standard output in Program VIBSYS.

Additional checks are inherent in the solution. In Option 1 the summation of the coefficient matrices of the $x(0) \cos \beta_r t$ terms must sum to the identity matrix, since the evaluation of x(t) at time zero must equal the initial conditions of displacement. This is easily seen by referring to equation 21 and considering the initial values of velocity to be zero. Similarly, in Option 2 the sum of the coefficient columns of the cosine terms must equal the initial conditions of displacement.

In the steady state solution with sinusoidal excitation, the inversion of the complex matrix presents one of the greatest possible sources of error. Therefore when Option 5 is executed, the output includes the real and imaginary parts of the inverse and the product of the original matrix and its inverse. This product should, of course, be the identity matrix.

3.3.2 External Checks. In order to determine the reliability of Program VIBSYS, sample problems of two degrees of freedom for which solutions were available in the literature were programmed. Correlation of results was excellent. No suitable sample problems involving more than two degrees of freedom were found in the literature. Therefore, in order to extend the range of testing, solutions of undamped systems were compared with program solutions of similar systems with negligible damping.

The results were as anticipated; as the damping was decreased the natural frequencies approached those of the undamped system. The results of a sample problem are demonstrated in Appendix E.

3.4 <u>Program Limitations</u>. Although the digital computer extends the size of system for which solutions are obtainable, the ultimate size is dictated by the storage capacity of the computer. The CDC 1604 computer has approximately 32,000 storage locations; however, it is easily seen that this may be rapidly exhausted. Program VIBSYS is limited to a system with 10 degrees of freedom.

The basic principle in the development of the general equations is the orthogonality relations of the eigenvectors and the parameters of the system. Therefore each eigenvalue must have associated with it a unique eigenvector. This requirement is not fulfilled in the case of a system having two equal eigenvalues, and therefore two natural frequencies of equal magnitude. In such a case VIBSYS determines the frequencies and the program terminates at that point.

If the damping is critical in any part of the system an attempt will be made to obtain the eigenvalues and eigenvectors. However, if they are found, they will not occur in the complex conjugate pairs and the program will be terminated.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

Program VIBSYS provides an accurate method for determining the natural frequencies, mode shapes, and time behavior of a subcritically damped dynamic system. The compiling time for the program is 4 minutes and 30 seconds, while the running time for a system with four degrees of freedom is approximately one minute. (Running time will depend on the number of iterations required to obtain the eigenvalues and eigenvectors.) Therefore it is more efficient, with respect to computer time, to make multiple runs.

Comparison of natural frequencies of undamped and lightly damped systems has shown that the undamped approximation is very good for damping ratios of less than 0.01. For systems tested, up to four degrees of freedom, the natural frequencies of the damped and undamped systems did not differ in the first three significant figures.

The program may be extended in a variety of possible ways. The general nature of Program VIBSYS requires liberal use of computer storage space. The size of the system may be extended by segmenting the program into separate programs for handling specific problems such as, free vibration, forced vibration with a general forcing function, or forced vibration with sinusoidal excitation. Furthermore, with minor modifications the present program may be terminated upon finding the natural frequencies and mode shapes and used as a separate program to obtain input data for specialized programs.

Program VIBSYS can be augmented and made more useful by eliminating the restrictions which cause the program to be terminated if either (a)

the system has two eigenvalues of equal magnitude, or (b) one of the modes has aperiodic free motion, that is a purely real eigenvalue.

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APPENDIX A

PROGRAM STRUCTURE

A.1 General Remarks. Program VIBSYS is composed of a main body which is divided into five sections, and two subroutines, INVERT and MATSUB. Function subroutines used include the sine, cosine, exponential, square root, and absolute value functions. These function subroutines are called from the Fortran library tape. Subroutine draw is also available on the Fortran library type and is programmed specifically for the CDC 1604 computer for use with the CALCOMP plotter.

Subroutine MATSUB, which is available as a CO-OP Library mathematical subroutine, required minor modifications in order to retain the eigenvalues and eigenvectors in storage for use in the main body of the program. The flow chart provided with MATSUB in the CO-OP Library was found to be inadequate for understanding. In order to perform the necessary modifications a revised flow chart was made up and included in Appendix C.

A.2 Main Body. The main body consists of five sections with functions as listed below

(a) Input - In addition to allocation of storage spaces for dimensioned arrays and setting up a control for reading input data, the input section contains a control sequence for MATSUB and curve labeling.

(b) Natural frequencies and mode shapes - The reduced characteristic matrix is constructed and the eigenvalues and eigenvectors are determined. The natural frequencies and mode shapes are then found.

(c) Forced Vibration - The coefficient column vectors of the sine and cosine terms of option 5 are calculated.

(d) Free Vibration - The outputs of options 1, 2, and 3 are formulated.

(e) Forced Vibration with General Excitation - The coefficient matrices of option 4 are calculated.

A.3 Subroutines.

(a) MATSUB - Evaluates the eigenvalues and eigenvectors of the reduced system.

(b) INVERT - Inverts a real matrix.

PROGRAM NOTATION

AB	-	coefficient column vector of $\cos\beta_r t$ term, output of option 2
ABC	-	coefficient column vector of $\cos\beta_r t$ term when X(0), and $\dot{X}(0)$ are modified to obtain total solution
ALIS	-	control parameter for MATSUB
ALRS	-	control parameter for MATSUB
AM	-	inverse of mass matrix
AMD	-	product of inverse of mass matrix and damping matrix
AMM	-	mass matrix lb-sec^2/in
AMS	-	product of inverse of mass matrix and stiffness matrix
ARN	-	$-2\alpha_r(V_r{}^TMV_r - W_r{}^TMW_r) - 4\beta_r \mid V_r{}^TMW_r + V_r{}^TCV_r - W_r{}^TCW_r$
ARY	-	imaginary part of identity matrix
BRN	-	$2\beta_r(V_r{}^TMV_r - W_r{}^TMW_r) - 4\alpha_r V_r{}^TMW_r + 2V_r{}^TCV_r$
CD	-	coefficient column vector of $\sin\beta_r t$ term, output of option 2
CDC	-	coefficient column vector of $\sin\beta_r t$ term when X(0) and $\dot{X}(0)$ are modified to obtain total solution
CPD	-	coefficient column of $\sin\omega t$, output of option 5
DI	-	imaginary part of driving force amplitude
DISPI	-	imaginary part of modal matrix
DISPR	-	real part of modal matrix
DM	-	damping matrix lb-sec/in
DR	-	real part of driving force amplitude
EP1	-	iteration parameter for MATSUB
EP2	-	iteration parameter for MATSUB

GBI - iteration parameter for MATSUB

GBR - iteration parameter for MATSUB

GRM - coefficient matrix of $\dot{X}(0) \cos \beta_r t$, output of option 1

HRM - coefficient matrix of $\dot{X}(0) \sin \beta_r t$, output of option 1

IC - control parameter for VIBSYS

IDET - control parameter for MATSUB

IEG - control parameter for MATSUB

IFC - control parameter for VIBSYS

ITITLE - title for graphical output

IVEC - control parameter for MATSUB

MIT - controls number of iterations in MATSUB, power method

MITS - controls number of iterations in MATSUB inverse power method

MP1 - option 1 control

MP2 - option 2 control

MP3 - option 3 control

MP4 - option 4 control

MP5 - option 5 control

N - order of system

RIN - real part of identity matrix

RMI - coefficient column vector of $\cos \omega t$ term, output of option 5

RPI - real part of the inverse of $[K - \omega^2 M + j \omega C]$

S - stiffness matrix lb/in

SPEC - spectral matrix

STEP - step size for graphical output

TIPI - imaginary part of inverse of $K - \omega^2 M + j \omega C$

UI - matrix of reduced system, imaginary part

UR - matrix of reduced system, real part

VALI - imaginary part of eigenvalue of reduced system (β_r)

VALR - real part of eigenvalue of reduced system (α_r)

VCO - initial velocity vector modified to account for initial velocity of steady state solution

VCV - $V_r^T C V_r$

VCW - $V_r^T C W_r$

VECI - imaginary part of eigenvector of reduced system (V_r)

VECR - real part of eigenvector of reduced system (W_r)

VO - initial velocity vector

VMV - $V_r^T M V_r$

VMW - $V_r^T M W_r$

WCW - $W_r^T C W_r$

WDM - imaginary part of $[K - \omega^2 M + j\omega C]$

WMK - real part of $[K - \omega^2 M + j\omega C]$

WMW - $W_r^T M W_r$

WO - excitation frequency

X - time coordinate for graphs

XCO - initial condition of displacement modified to account for initial displacement of steady state solution

XO - initial displacement vector

XOC - coefficient matrix of X(0) $\cos \beta_r t$ term, output of option 1

XOS - coefficient matrix of X(0) $\sin \beta_r t$ term, output of option 2

Y - ordinate of graphical output

The above notation lists the principal array and parameter names used in the main body of the program. Array names not listed are used for intermediate operations. Where appropriate the names are associated with the

symbols used in the mathematical analysis.

APPENDIX B

INSTRUCTIONS FOR USE OF PROGRAM

B.1 Purpose. The purpose of Program VIBSYS is to determine the natural frequencies, mode shapes, and time behavior of a subcritically damped, linear dynamic system with N degrees of freedom. ($N \lesssim 10$) The mass, damping and stiffness matrices of the system must be known.

B.2 Input Data. A blank card must follow the last end card of the program deck. The data cards then follow in the order and format listed below. The first data card uses the input format 9I5 and is the control card for the program. The nine fields are designated as follows:

1.	N	=	the order of the system
2.	MP1	=	1, Option 1 executed 0, Omit
3.	MP2	=	1, Option 2 executed 0, Omit
4.	MP3	=	1, Option 3 executed 0, Omit
5.	MP4	=	1, Option 4 executed 0, Omit
6.	MP5	=	1, Option 5 executed 0, Omit
7.	IC	=	1, Initial conditions given 0, Initial conditions are zero
8.	IFC	=	1, Amplitude and frequency of excitation given 0, Omit
9.	NPTS		Number of points to be calculated for graphical output

All nine fields must be right justified.

The control card is followed by the mass, damping, and stiffness matrices respectively. Each matrix is read in rowwise using input format 8F10.3* with each row starting on a new card for systems with $N \lesssim 8$. For systems with $N > 8$ the elements are read in rowwise in a sequential manner, that is the first element of the second row should appear in the field adjacent to the last element of the first row. In this case each matrix must start on a new card.

The initial displacement vector, initial velocity vector, real part of the driving force amplitude vector, and imaginary part of the driving force amplitude vector, follow, in that order, using format 8F10.3.* Each vector must start on a new card.

The final data card uses a 2F10.3* format. The first field contains the excitation frequency and the second the step size (i.e., time increment) for the graphical output option.

Blank cards must be used for initial conditions of displacement and velocity, real and imaginary parts of forcing function amplitudes and final data card when these values are zero. Multiple runs may be processed by placing additional data decks behind the first. The program is terminated by placing a blank card behind the last data card.

---

* Should this format present undesirable restrictions on the size of the elements of the matrix the so called "E" field may be used by changing card number 32. The "E" field was avoided since it is more susceptible to error by the user.

B.3 Deck Assembly. The first card of the program deck is a job card. The statement "Use scratch tape" must be included in addition to the required job card information.

```
JOB CARD
┌PROGRAM VIBSYS
│
│
│
└END
┌SUBROUTINE INVERT
│
│
│
└END
┌SUBROUTINE MATSUB
│
│
│
│END
└END
BLANK CARD
┌DATA CARDS
│
│
└
BLANK CARD
```

B.4 Cautions to User. The curves drawn in the graphical output option are straight line approximations between computed values. The step size must therefore be chosen appropriately in order to obtain a smooth curve. Since the maximum number of points permitted by subroutine DRAW is 900, it

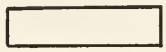
may not be possible to see the transient phase completely die out.

If difficulty is encountered in obtaining the eigenvalues and eigenvectors, the control parameters of MATSUB may be altered to enhance convergence. The value of MIT and MITS, card numbers 20 and 21, control the maximum number of iterations to be performed in the power method and inverse power method respectively. ALRS and ALIS, card numbers 78 and 79, represent a complex parameter that may be called the "origin". MATSUB will usually converge on the eigenvalue most distant from the "origin". ALRS = 1.0 and ALIS = 0.1 in Program VIBSYS. The convergence criteria for the power method is set by the value of EP1, card number 85 while the inverse power method is determined by EP2 card number 86. EP1 = 10^{-4}, and EP2 = 10^{-14} in the present version. Furthermore, if IEG, card number 76, is set equal to one, the eigenvalue iterants will be printed out and more suitable values of ALRS and ALIS may possibly be found by inspection.

APPENDIX C

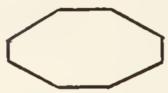
FLOW CHARTS

The symbols used in the flow charts are defined as follows:

 - arithmetic, read or print operations

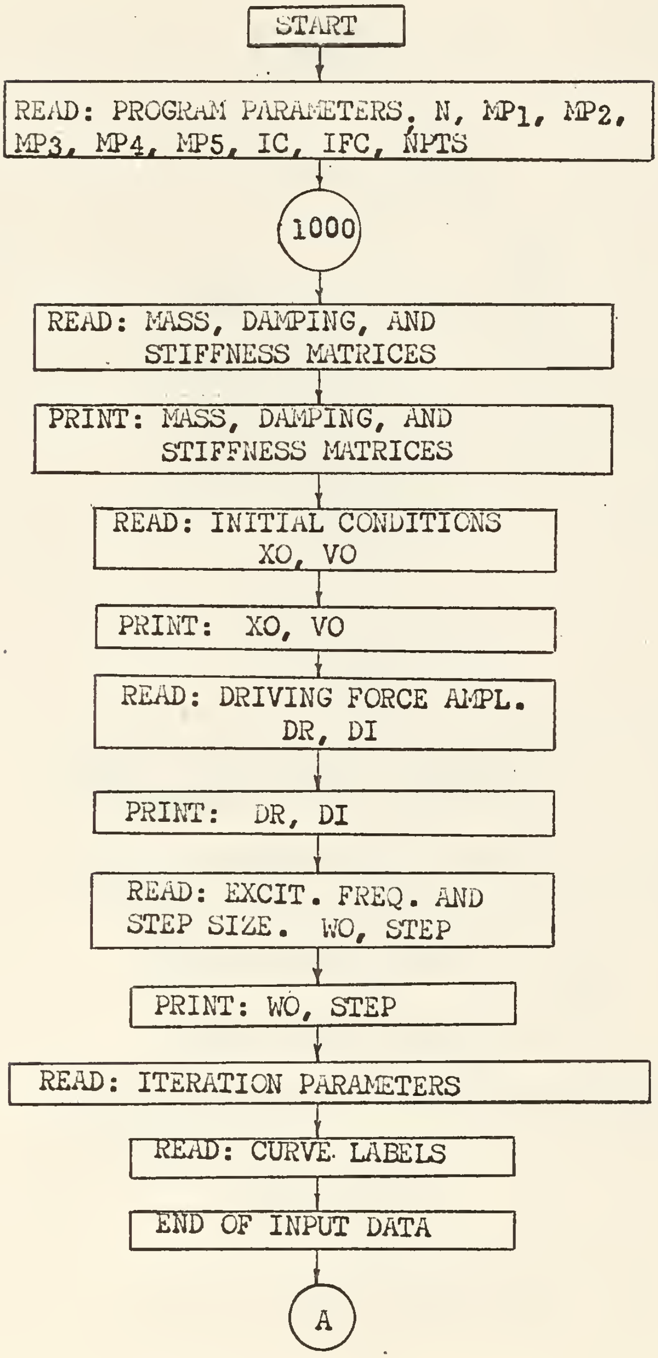
 - connectors

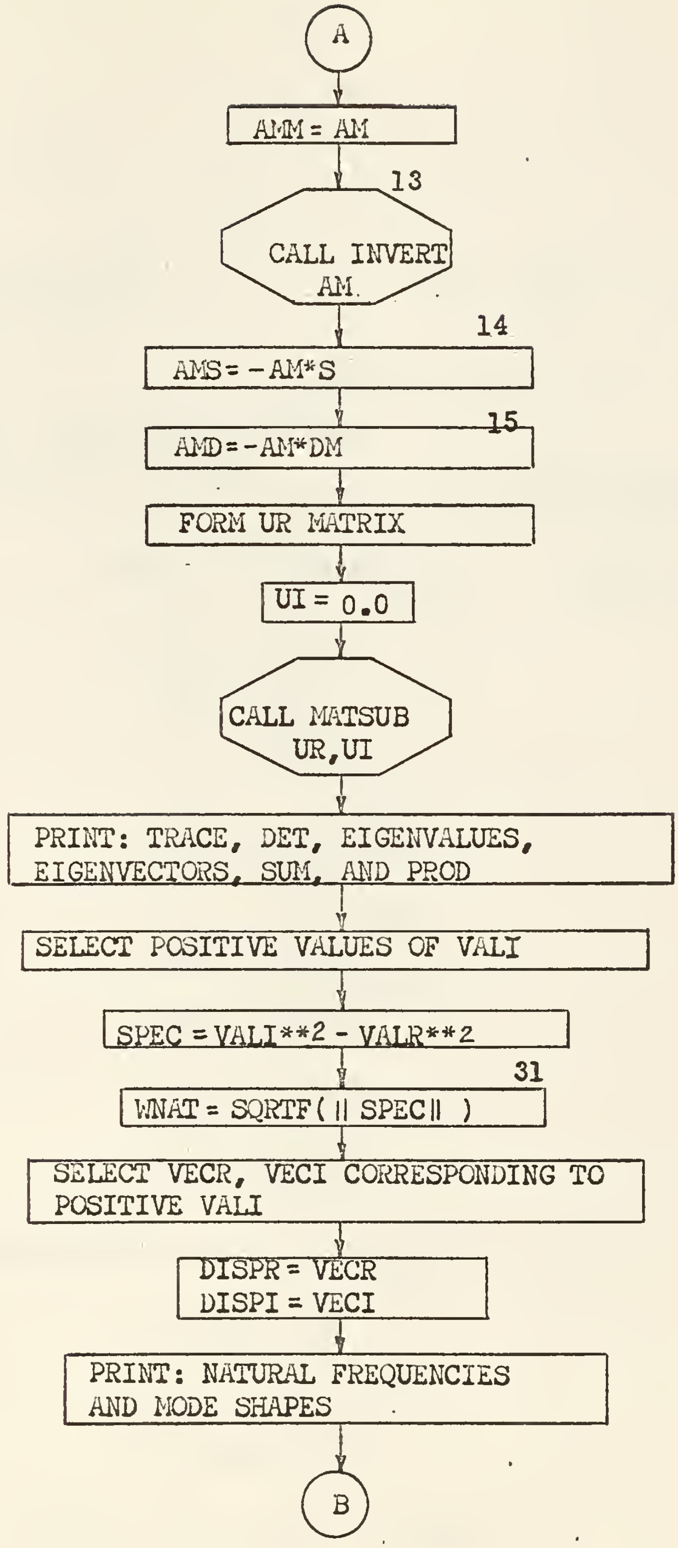
 - decision

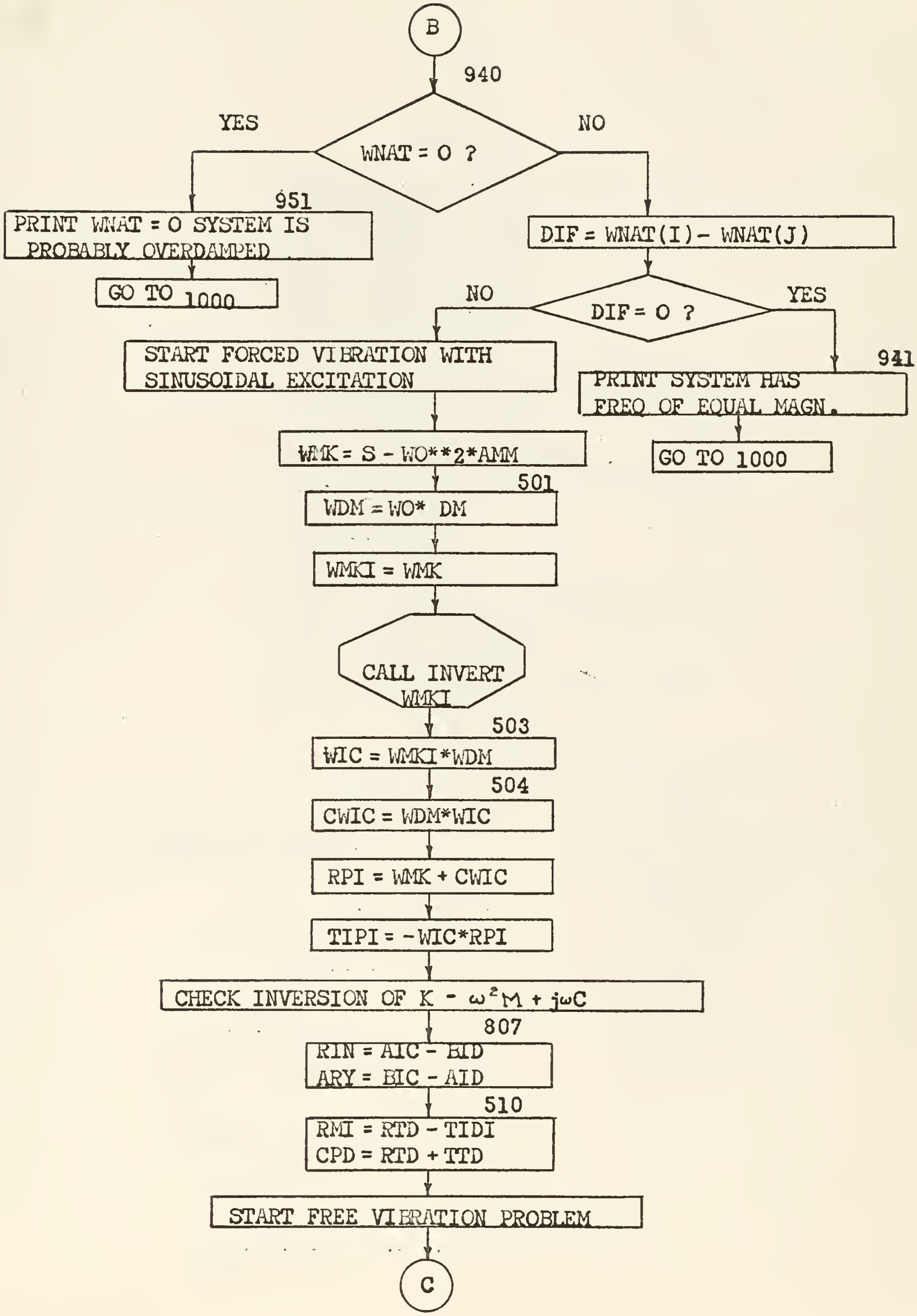
 - call subroutine indicated

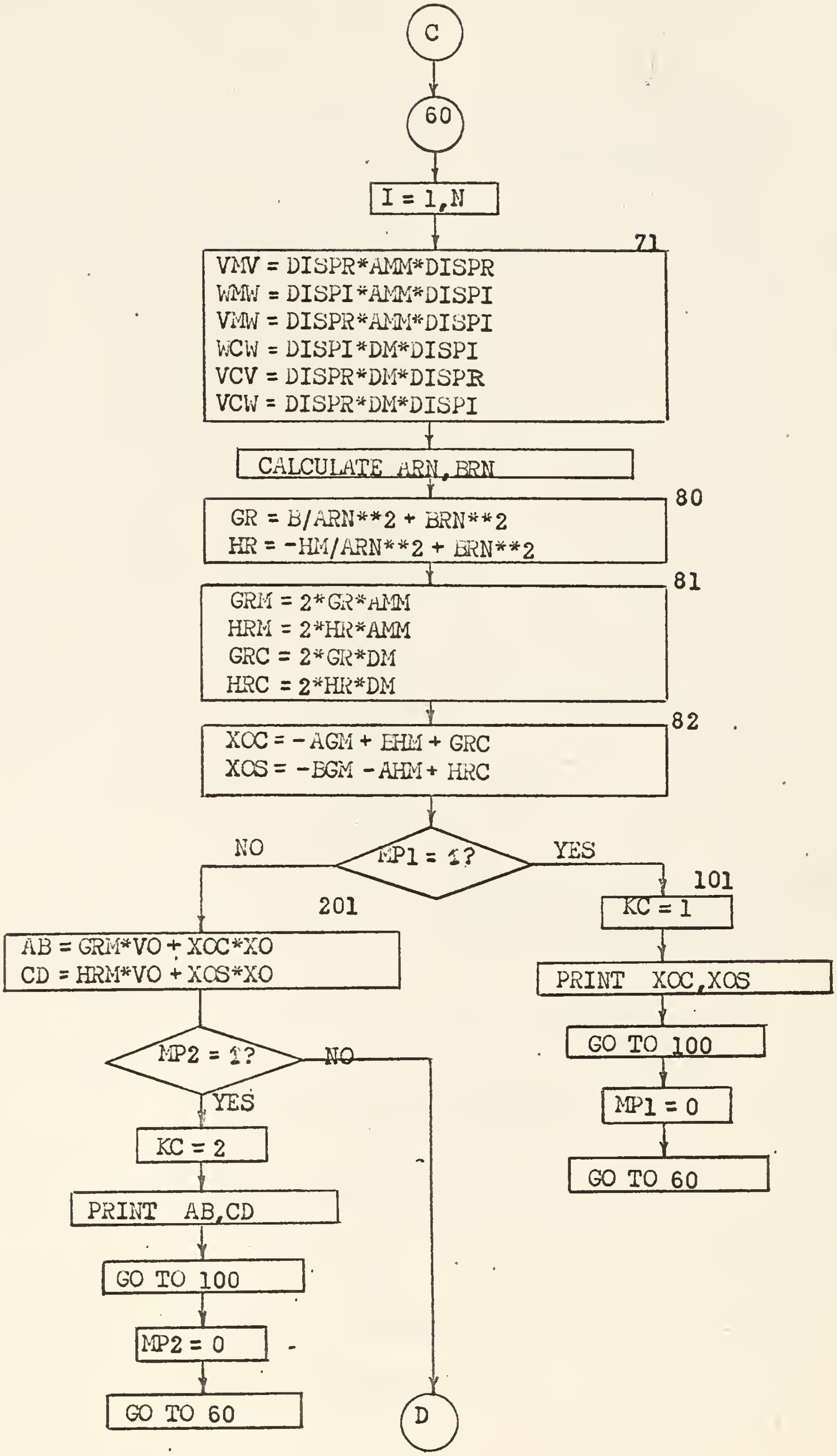
The notation used in the flow chart for MATSUB corresponds to that used in the CO OP Library version, reference 7.

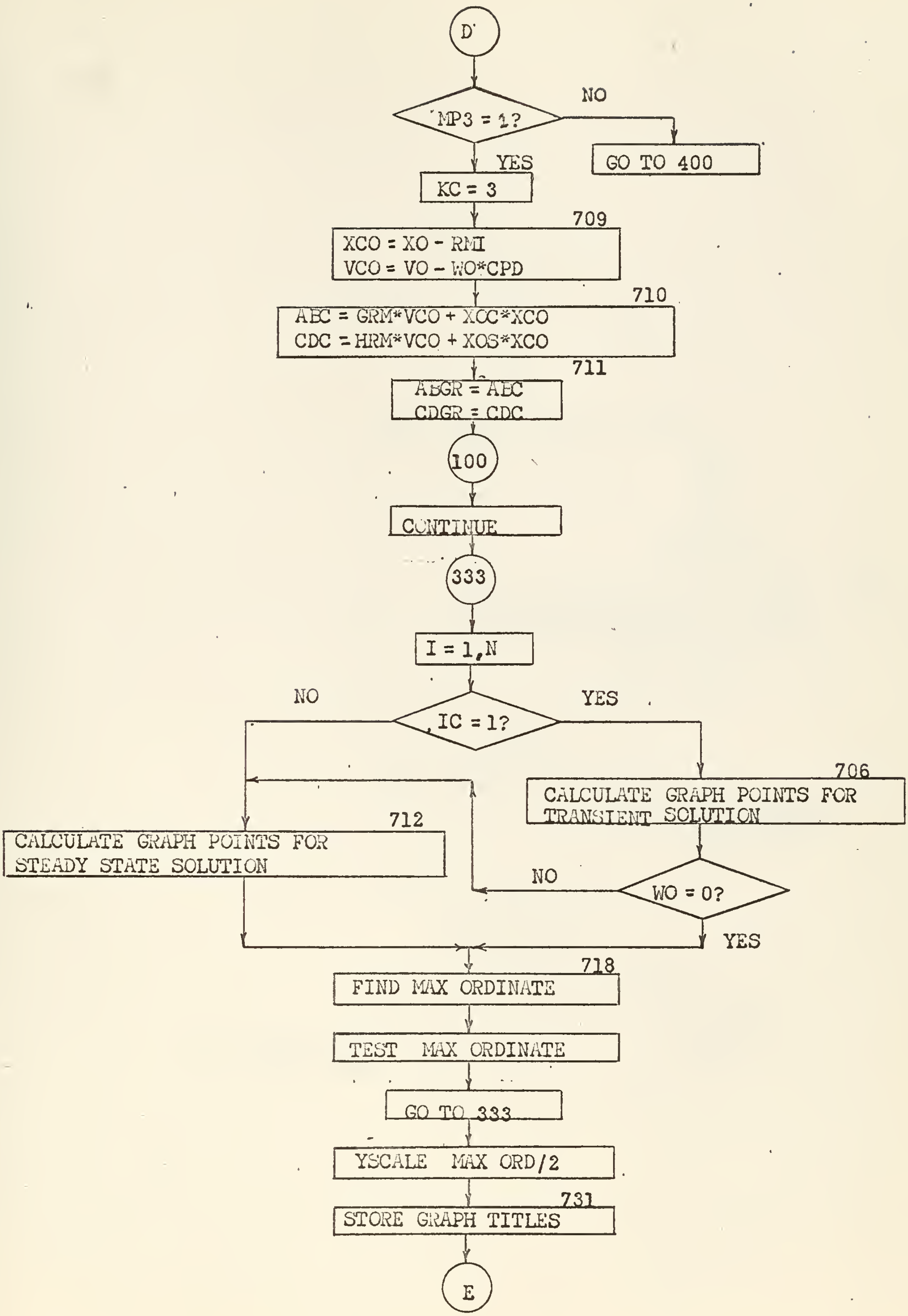
FLOW CHART MAIN BODY PROGRAM VIBSYS

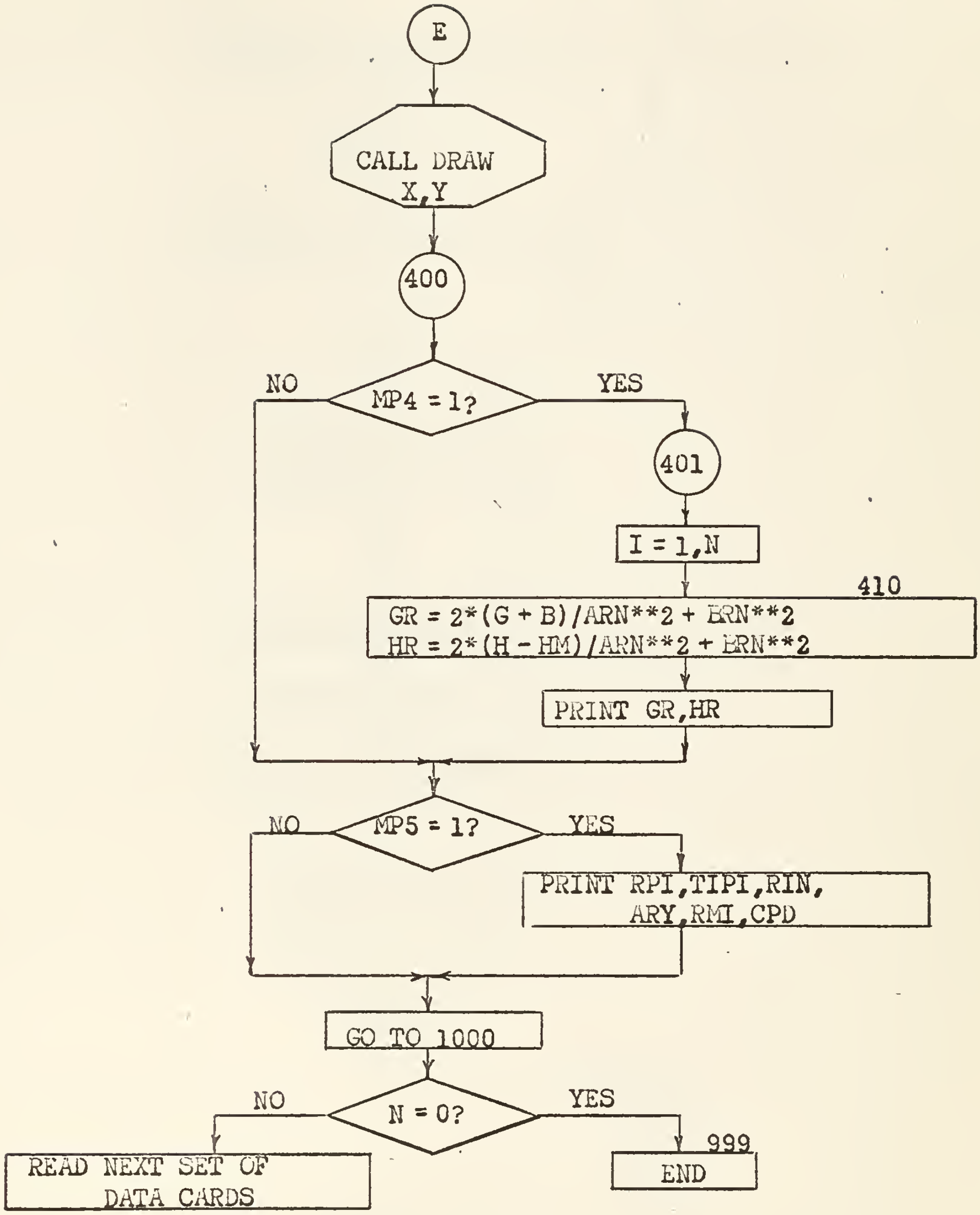



A
AMM = AM
13
CALL INVERT AM
14
AMS = -AM*S
15
AMD = -AM*DM
FORM UR MATRIX
UI = 0.0
CALL MATSUB UR,UI
PRINT: TRACE, DET, EIGENVALUES, EIGENVECTORS, SUM, AND PROD
SELECT POSITIVE VALUES OF VALI
SPEC = VALI**2 - VALR**2
31
WNAT = SQRTF(|| SPEC ||)
SELECT VECR, VECI CORRESPONDING TO POSITIVE VALI
DISPR = VECR
DISPI = VECI
PRINT: NATURAL FREQUENCIES AND MODE SHAPES
B

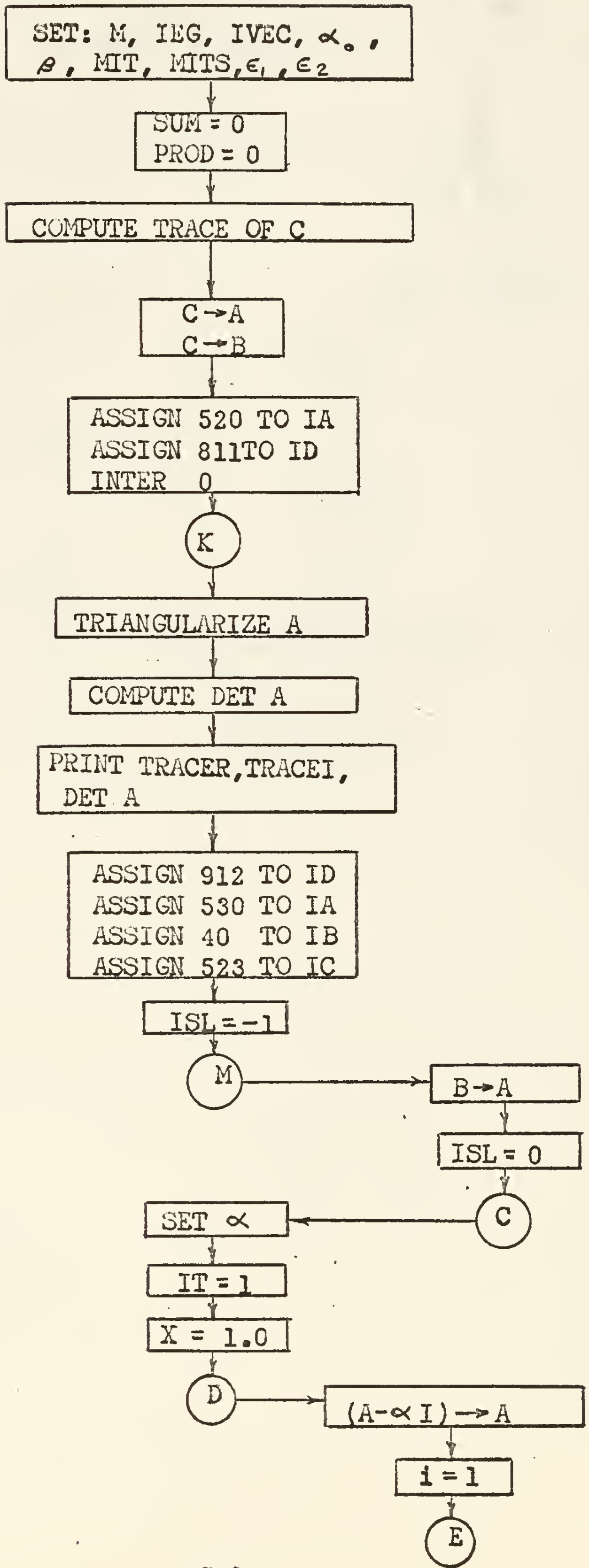

B
940
YES
NO
WNAT = 0 ?
951
PRINT WNAT = 0 SYSTEM IS PROBABLY OVERDAMPED
GO TO 1000
DIF = WNAT(I) - WNAT(J)
NO
YES
DIF = 0 ?
941
START FORCED VIBRATION WITH SINUSOIDAL EXCITATION
PRINT SYSTEM HAS FREQ OF EQUAL MAGN.
WMK = S - WO**2*AMM
GO TO 1000
501
WDM = WO* DM
WMKI = WMK
CALL INVERT WMKI
503
WIC = WMKI*WDM
504
CWIC = WDM*WIC
RPI = WMK + CWIC
TIPI = -WIC*RPI
CHECK INVERSION OF K - ω²M + jωC
807
RIN = AIC - BID
ARY = BIC - AID
510
RMI = RTD - TIDI
CPD = RTD + TTD
START FREE VIBRATION PROBLEM
C

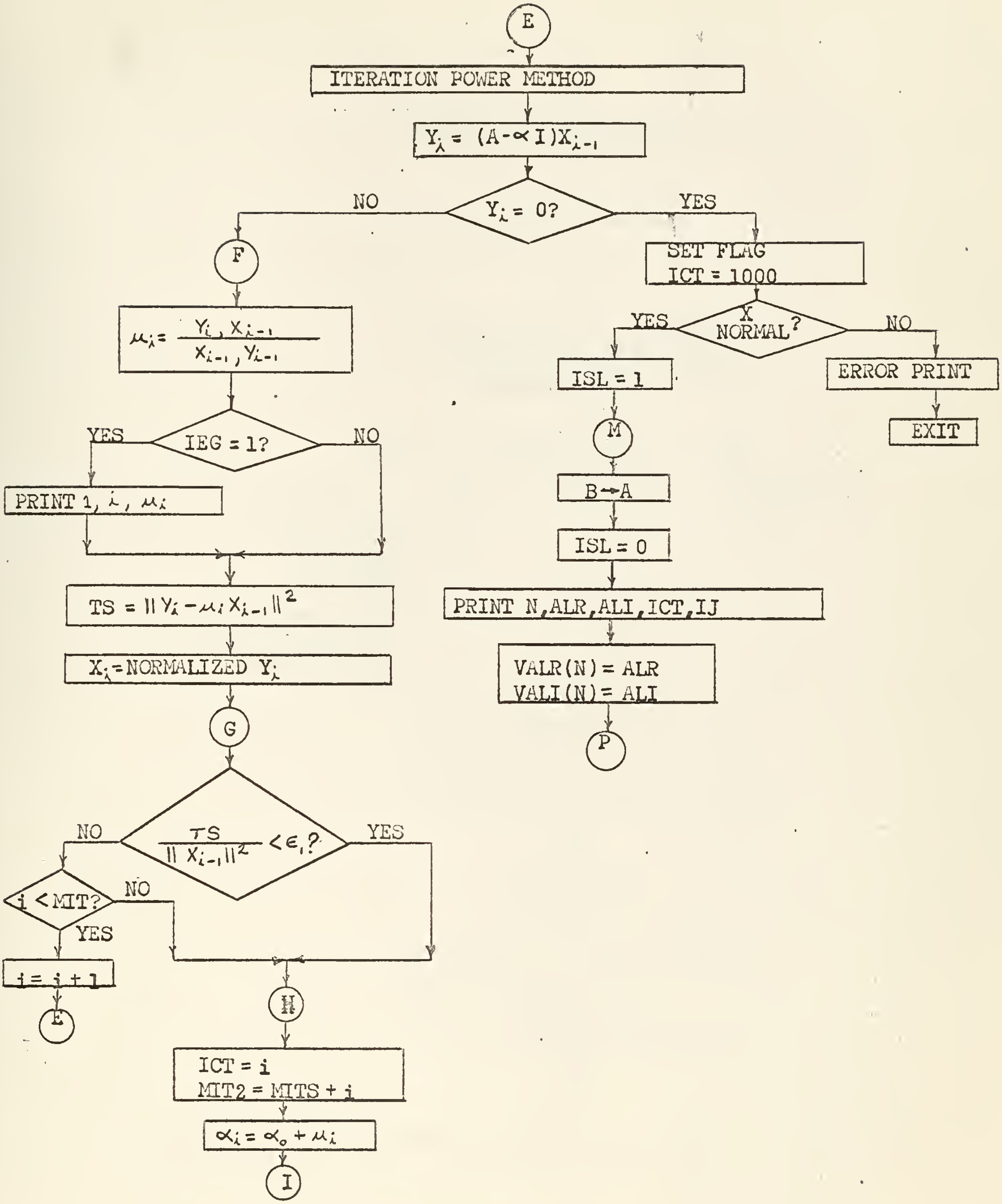

C
60
I = 1,N
71
VMV = DISPR*AMM*DISPR
WMW = DISPI*AMM*DISPI
VMW = DISPR*AMM*DISPI
WCW = DISPI*DM*DISPI
VCV = DISPR*DM*DISPR
VCW = DISPR*DM*DISPI
CALCULATE ARN, BRN
80
GR = B/ARN**2 + BRN**2
HR = -HM/ARN**2 + BRN**2
81
GRM = 2*GR*AMM
HRM = 2*HR*AMM
GRC = 2*GR*DM
HRC = 2*HR*DM
82
XOC = -AGM + BHM + GRC
XOS = -BGM - AHM + HRC
NO
MP1 = 1?
YES
201
AB = GRM*VO + XOC*XO
CD = HRM*VO + XOS*XO
MP2 = 1?
NO
YES
KC = 2
PRINT AB,CD
GO TO 100
MP2 = 0
GO TO 60
D
101
KC = 1
PRINT XOC,XOS
GO TO 100
MP1 = 0
GO TO 60

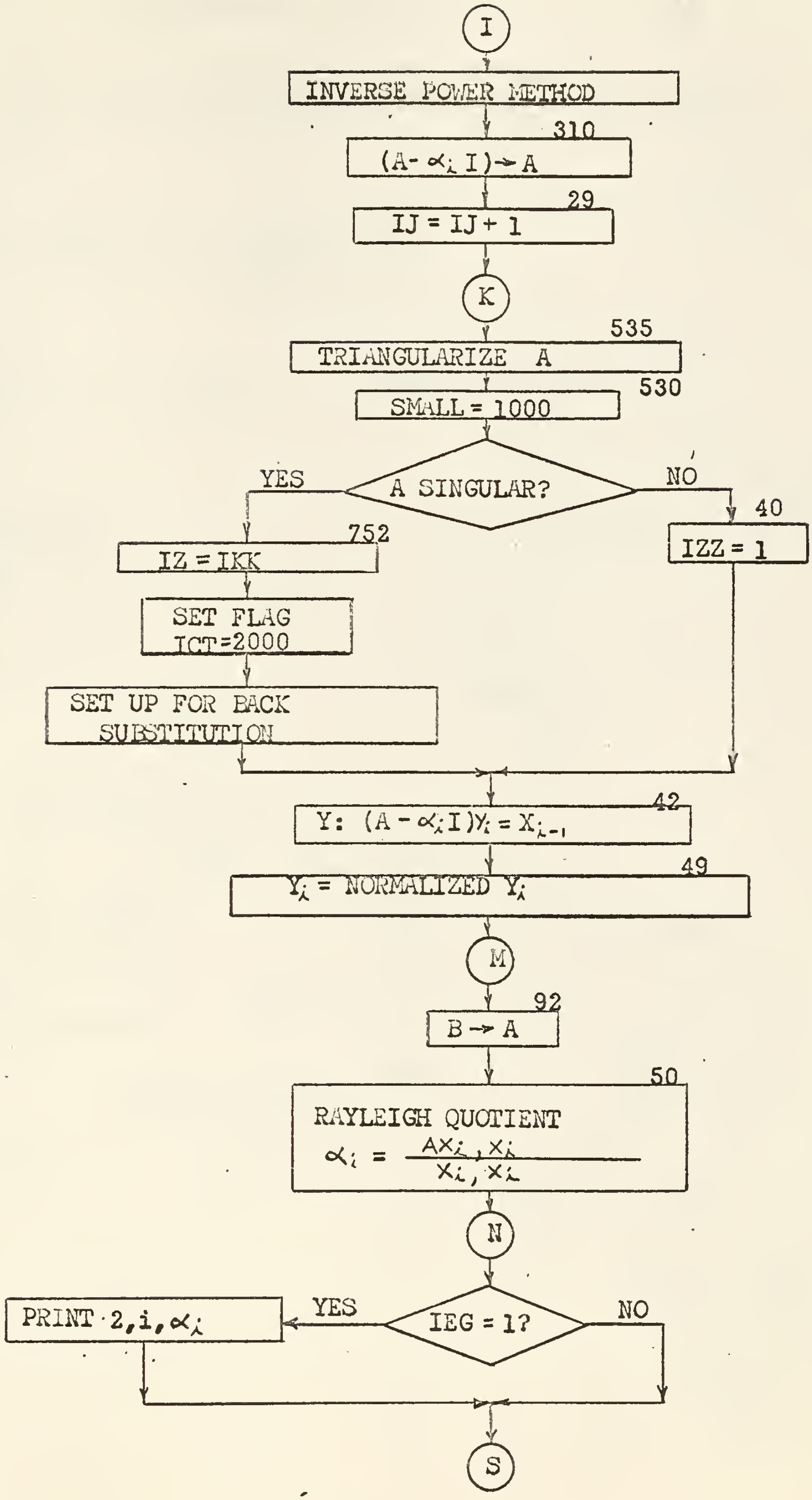

D
MP3 = 1?
NO
GO TO 400
YES
KC = 3
709
XCO = XO - RMI
VCO = VO - WO*CPD
710
ABC = GRM*VCO + XOC*XCO
CDC = HRM*VCO + XOS*XCO
711
ABGR = ABC
CDGR = CDC
100
CONTINUE
333
I = 1,N
NO
YES
IC = 1?
706
CALCULATE GRAPH POINTS FOR TRANSIENT SOLUTION
712
CALCULATE GRAPH POINTS FOR STEADY STATE SOLUTION
NO
WO = 0?
YES
718
FIND MAX ORDINATE
TEST MAX ORDINATE
GO TO 333
YSCALE MAX ORD/2
731
STORE GRAPH TITLES
E

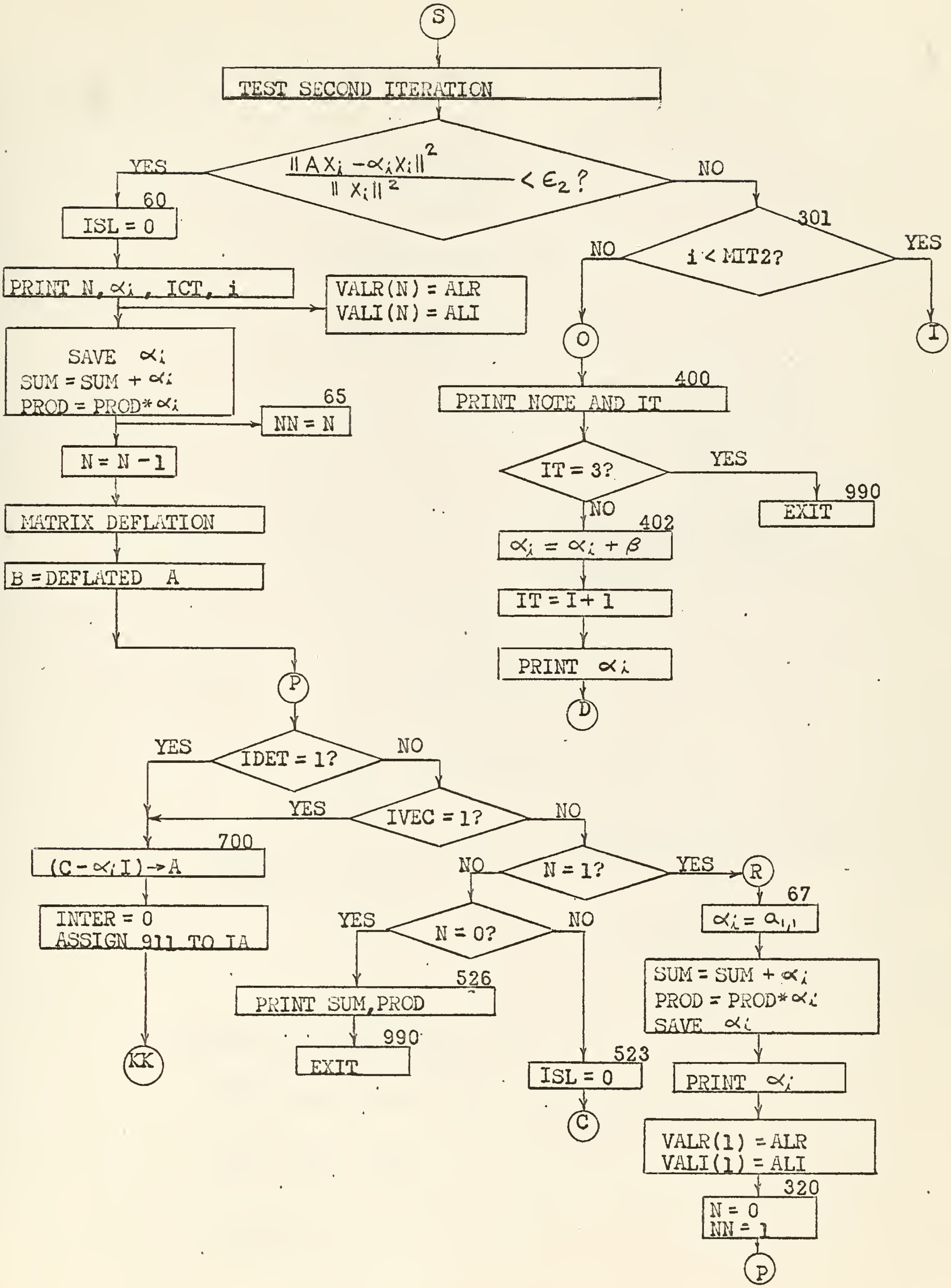

E
CALL DRAW X,Y
400
NO
MP4 = 1?
YES
401
I = 1,N
410
GR = 2*(G + B)/ARN**2 + BRN**2
HR = 2*(H - HM)/ARN**2 + BRN**2
PRINT GR,HR
NO
MP5 = 1?
YES
PRINT RPI,TIPI,RIN, ARY,RMI,CPD
GO TO 1000
NO
N = 0?
YES
READ NEXT SET OF DATA CARDS
999
END

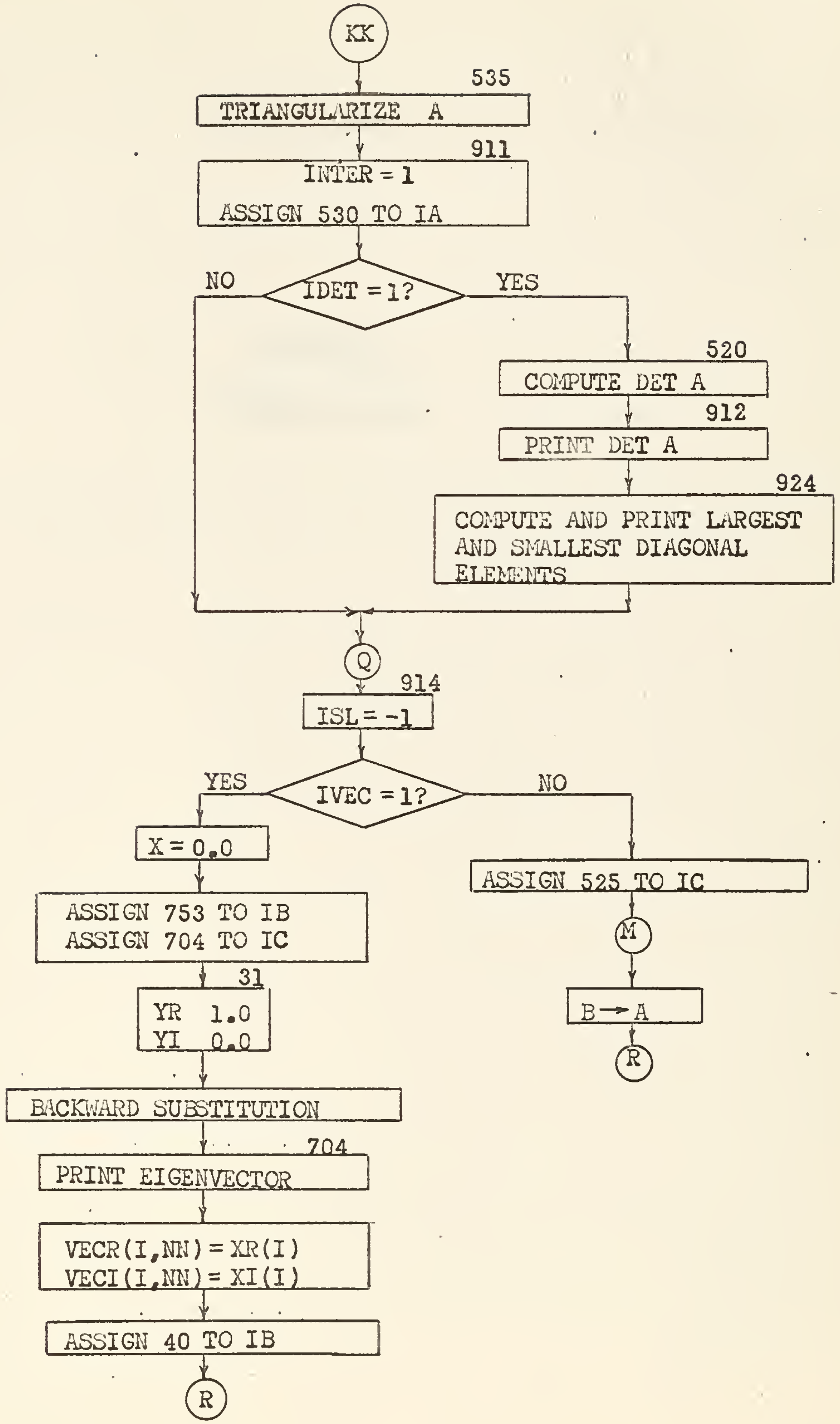
SUBROUTINE MATSUB




E
ITERATION POWER METHOD
$Y_i = (A-\alpha I)X_{i-1}$
$Y_i = 0?$
NO
YES
SET FLAG
ICT = 1000
X NORMAL?
YES
NO
ISL = 1
ERROR PRINT
EXIT
M
B→A
ISL = 0
PRINT N,ALR,ALI,ICT,IJ
VALR(N) = ALR
VALI(N) = ALI
P
F
$\mu_i = \frac{Y_i, X_{i-1}}{X_{i-1}, Y_{i-1}}$
IEG = 1?
YES
NO
PRINT 1, i, μ_i
$TS = \| Y_i - \mu_i X_{i-1} \|^2$
X_i = NORMALIZED Y_i
G
$\frac{TS}{\| X_{i-1} \|^2} < \epsilon_1 ?$
NO
YES
i < MIT?
NO
YES
i = i + 1
E
H
ICT = i
MIT2 = MITS + i
$\alpha_i = \alpha_0 + \mu_i$
I


I
INVERSE POWER METHOD
310
(A- αi I)→A
29
IJ = IJ + 1
K
535
TRIANGULARIZE A
530
SMALL = 1000
A SINGULAR?
YES
NO
752
IZ = IKK
SET FLAG
ICT=2000
SET UP FOR BACK
SUBSTITUTION
40
IZZ = 1
42
Y: (A - αi I)Yi = Xi-1
49
Yi = NORMALIZED Yi
M
92
B → A
50
RAYLEIGH QUOTIENT
αi = Axi, xi / xi, xi
N
IEG = 1?
YES
NO
PRINT 2,i,αi
S


S
TEST SECOND ITERATION
$\frac{\| AX_i - \alpha_i X_i \|^2}{\| X_i \|^2} < \epsilon_2$?
YES
NO
60
ISL = 0
PRINT N, α_i, ICT, i
VALR(N) = ALR
VALI(N) = ALI
SAVE α_i
SUM = SUM + α_i
PROD = PROD* α_i
65
NN = N
N = N − 1
MATRIX DEFLATION
B = DEFLATED A
P
301
i < MIT2?
NO
YES
O
I
400
PRINT NOTE AND IT
IT = 3?
YES
990
EXIT
NO
402
$\alpha_i = \alpha_i + \beta$
IT = I + 1
PRINT α_i
D
IDET = 1?
YES
NO
IVEC = 1?
YES
NO
700
$(C - \alpha_i I) \rightarrow A$
INTER = 0
ASSIGN 911 TO IA
KK
N = 1?
NO
YES
R
67
$\alpha_i = a_{1,1}$
N = 0?
YES
NO
526
PRINT SUM, PROD
990
EXIT
523
ISL = 0
C
SUM = SUM + α_i
PROD = PROD* α_i
SAVE α_i
PRINT α_i
VALR(1) = ALR
VALI(1) = ALI
320
N = 0
NN = 1
P


KK
535
TRIANGULARIZE A
911
INTER = 1
ASSIGN 530 TO IA
NO
IDET = 1?
YES
520
COMPUTE DET A
912
PRINT DET A
924
COMPUTE AND PRINT LARGEST AND SMALLEST DIAGONAL ELEMENTS
Q
914
ISL = -1
YES
IVEC = 1?
NO
X = 0.0
ASSIGN 753 TO IB
ASSIGN 704 TO IC
31
YR 1.0
YI 0.0
BACKWARD SUBSTITUTION
704
PRINT EIGENVECTOR
VECR(I,NN) = XR(I)
VECI(I,NN) = XI(I)
ASSIGN 40 TO IB
R
ASSIGN 525 TO IC
M
B → A
R

APPENDIX D

PROGRAM LISTING

```
..JOB0199F, MIKLOS T J  USE SCRATCH TAPE
      PROGRAM VIBSYS
     0DIMENSION AM(10,10),DM(10,10),S(10,10),AMS(10,10),AMD(10,10),VALR
     1(20),VALI(20),VECR(20,20),VECI(20,20),AR(20,20),AI(20,20),BR(20,20
     2),BI(20,20),UR(20,20),UI(20,20),XR(20),XI(20),YR(20),YI(20),ZR(20)
     3,ZI(20)
     0COMMON AM,DM,S,AMS,AMD,VALR,VALI,VECR,VECI,AR,AI,BR,BI,UR,UI,XR,XI
     1,YR,YI,ZR,ZI
     0DIMENSION SPEC(10,10),DISPR(10,10),DISPI(10,10),AMM(10,10),VMV(10)
     1,WMW(10),VMW(10),VCV(10),WCW(10),VCW(10),ARN(10),BRN(10),G(10,10),
     2B(10,10),GR(10,10),H(10,10),HM(10,10),HR(10,10),GRM(10,10),HRM(10,
     310),GRC(10,10),HRC(10,10),AGM(10,10),BGM(10,10),BHM(10,10),AHM(10,
     410),XOC(10,10),XOS(10,10),VAR(10),VAI(10),XO(10),VO(10),AB(10),CD
     5(10),X(900),Y(900),ITITLE(12),WMK(10,10),WMKI(10,10),WDM(10,10),
     6WIC(10,10),CWIC(10,10),RPI(10,10),TIPI(10,10),DR(10),DI(10),RTD(10
     7),ITD(10),ITDI(10),RMI(10),CPD(10),RTDI(10),WNAT(10),BIC(10,10),
     8AVAR(10),TEST(10),LAB(10),XCO(10),VCO(10),ABC(10),CDC(10),ABGR(10,
     910),CDGR(10,10)
      DIMENSION AIC(10,10),BID(10,10),AID(10,10),RIN(10,10),ARY(10,10)
 1000 READ 1,N,MP1,MP2,MP3,MP4,MP5,IC,IFC,NPTS
    1 FORMAT(9I5)
      REWIND 6
      IF(N)998,998,750
  750 PRINT 700
  7000FORMAT (16H1 PROGRAM VIBSYS,36X,20HT. J. MIKLOS     NHA2,40X,8HMAY
     11965//,20X,83HMATRIX ANALYSIS OF A MULTI-DEGREE OF FREEDOM VIBRATI
     2ON SYSTEM WITH VISCOUS DAMPING.////)
      PRINT 2,N
    2 FORMAT(20H      SYSTEM OF ORDER I5)
      DO 3 I=1,N
    3 READ 4,(AM(I,J),J=1,N)
    4 FORMAT(8F10.3)
  211 FORMAT(6E20.3)
      PRINT 5
    5 FORMAT(19H0     INERTIA MATRIX)
      DO 6 I= 1,N
    6 PRINT 211,(AM(I,J),J=1,N)
      DO 7 I= 1,N
    7 READ 4,(DM(I,J),J=1,N)
      PRINT 8
    8 FORMAT (19H0     DAMPING MATRIX)
      DO 9 I= 1,N
    9 PRINT 211,(DM(I,J),J=1,N)
      DO 10 I=1,N
   10 READ 4,(S(I,J),J=1,N)
      PRINT 11
   11 FORMAT(21H0     STIFFNESS MATRIX)
      DO 12 I=1,N
   12 PRINT 211,(S(I,J),J=1,N)
  210 READ 4,(XO(I),I=1,N)
      PRINT 751
  751 FORMAT(31H0 THE INITIAL DISPLACEMENTS ARE//)
      PRINT 211 (XO(I),I=1,N)
  212 READ 4,(VO(I),I=1,N)
      PRINT 752
  752 FORMAT(40H0 THE INITIAL CONDITIONS OF VELOCITY ARE//)
      PRINT 211 (VO(I),I=1,N)
  551 READ 4,(DR(I),I=1,N)
      READ 4,(DI(I),I=1,N)
      PRINT 753
  753 FORMAT(53H0 THE REAL PART OF THE AMPLITUDE OF THE DRIVING FORCE//)
      PRINT 211 (DR(I),I=1,N)
```

```
      PRINT 754
  754 FORMAT(53H0 THE IMAG PART OF THE AMPLITUDE OF THE DRIVING FORCE//)
      PRINT 211(DI(I),I=1,N)
      READ 652,WO,STEP
  652 FORMAT(2F10.5)
      PRINT 755, WO
  7550FORMAT(43H0 THE FREQUENCY OF THE FORCING FUNCTION IS F10.4,18HRADI
     1ANS PER SECOND//)
      PRINT 756,STEP
  7560FORMAT(44H0 THE STEP SIZE FOR THE GRAPHICAL OPTION IS F5.4,7HSECON
     1DS//,10X,17HEND OF INPUT DATA///)
C     ITERATION PARAMETERS
  213 M=2*N
      IEG =0
      IVEC =1
      ALRS =1.0
      ALIS =0.1
      GBR = 0.1
      GBI = 0.1
      IDET = 0
      MIT = 20
      MIIS = 20
      EP1  = 1.E-4
      EP2 = 1.E-14
      LAB(1)=4H M1
      LAB(2)=4H M2
      LAB(3)=4H M3
      LAB(4)=4H M4
      LAB(5)=4H M5
      LAB(6)=4H M6
      LAB(7)=4H M7
      LAB(8)=4H M8
      LAB(9)=4H M9
      LAB(10)=4H M10
C     END OF INPUT DATA
      DO 58 I=1,N
      DO 58 J=1,N
   58 AMM(I,J)=AM(I,J)
   13 CALL INVERT (AM,N,D)
      DO 14 I=1,N
      DO 14 J=1,N
      AMS(I,J)=0.0
      DO 14 K=1,N
   14 AMS(I,J)=AMS(I,J)-AM(I,K)*S(K,J)
      DO 15 I=1,N
      DO 15 J=1,N
      AMD(I,J)=0.0
      DO 15 K=1,N
   15 AMD(I,J)=AMD(I,J)-AM(I,K)*DM(K,J)
C     FORM UR MATRIX
      DO 161 K=1,M
      DO 161 L=1,M
  161 UR(K,L)=0.0
      DO 16 K=1,N
      L=N+K
   16 UR(K,L)=1.0
      DO 17 I=1,N
      DO 17 J=1,N
      K=N+I
      L=J
   17 UR(K,L)=AMS(I,J)
      DO 18 I=1,N
```

```
      DO 18 J=1,N
      K=N+I
      L=N+J
   18 DR(K,L)=AMD(I,J)
      DO 170 I=1,M
      DO 170 J=1,M
  170 DI(I,J)=0.0
   19 CALL MATSUB (M,IEG,IVEC,ALRS,ALIS,GBR,GBI,IDET,MIT,MITS,EP1,EP2)
      L=0
      K=0
      DO 50 J=1,M
      IF(VALI(J))50,50,30
   30 K=K+1
      VAR(K)=VALR(J)
      VAI(K)=VALI(J)
      SPEC(K,K)=VALI(J)**2-VALR(J)**2
      DO 31 JJ=1,N
   31 WNAT(JJ)=SQRTF(ABSF(SPEC(JJ,JJ)))
      DO 32 I=1,M
      IF(ABSF(VECR(I,J))-1.0)32,35,32
   32 CONTINUE
   35 L=L+1
      IF(I-M/2)36,36,38
   36 DO 40 II=1,N
      DISPR(II,L)=VECR(II,L)
   40 DISPI(II,L)=VECI(II,L)
      GO TO 50
   38 DO 41 II=1,N
      DISPR(II,L)=VECR(II+N,J)
   41 DISPI(II,L)=VECI(II+N,J)
   50 CONTINUE
      PRINT 51
   51 FORMAT (30H0  THE NATURAL FREQUENCIES ARE)
   52 PRINT 53,(WNAT(J),J=1,N)
   53 FORMAT (4E20.8)
      PRINT 54
   54 FORMAT (28H0    MODAL MATRIX, REAL PART)
      DO 55 I=1,N
   55 PRINT 53,(DISPR(I,J),J=1,N)
      PRINT 56
   56 FORMAT (28H0    MODAL MATRIX, IMAG PART)
      DO 57 I=1,N
   57 PRINT 53,(DISPI(I,J),J=1,N)
      NM1=N-1
      DO 940 I=1,N
      IF(WNAT(I)-0.0001)951,951,940
  940 CONTINUE
      DO 950 I=1,NM1
      IP1=I+1
      DO 950 J=IP1,N
      DIF=WNAT(I)-WNAT(J)
      IF(ABSF(DIF)-0.0001)941,941,950
  950 CONTINUE
C     START FORCED VIBRATION WITH SINUSOIDAL EXCITATION
      DO 501 I=1,N
      DO 501 J=1,N
      WMK(I,J)=S(I,J)-WO**2*AMM(I,J)
  501 WDM(I,J)=WO*DM(I,J)
      DO 502 I=1,N
      DO 502 J=1,N
  502 WMKI(I,J)=WMK(I,J)
      CALL INVERT(WMKI,N,D)
```

```
      DO 503 I=1,N
      DO 503 J=1,N
      WIC(I,J)=0.0
      DO 503 K=1,N
  503 WIC(I,J)=WIC(I,J)+WMKI(I,K)*WDM(K,J)
      DO 504 I=1,N
      DO 504 J=1,N
      CWIC(I,J)=0.0
      DO 504 K=1,N
  504 CWIC(I,J)=CWIC(I,J)+WDM(I,K)*WIC(K,J)
      DO 505 I=1,N
      DO 505 J=1,N
  505 RPI(I,J)=WMK(I,J)+CWIC(I,J)
      CALL INVERI(RPI,N,D)
      DO 506 I=1,N
      DO 506 J=1,N
      TIPI(I,J)=0.0
C     CHECK INVERSION OF (K-W**2M+JWC)
      DO 506 K=1,N
  506 TIPI(I,J)=TIPI(I,J)-WIC(I,K)*RPI(K,J)
      DO 806 I=1,N
      DO 806 J=1,N
      AIC(I,J)=0.0
      BID(I,J)=0.0
      BIC(I,J)=0.0
      AID(I,J)=0.0
      DO 806 K=1,N
      AIC(I,J)=AIC(I,J)+WMK(I,K)*RPI(K,J)
      BID(I,J)=BID(I,J)+WDM(I,K)*TIPI(K,J)
      BIC(I,J)=BIC(I,J)+WDM(I,K)*RPI(K,J)
  806 AID(I,J)=AID(I,J)+WMK(I,K)*TIPI(K,J)
      DO 807 I=1,N
      DO 807 J=1,N
      RIN(I,J)=AIC(I,J)-BID(I,J)
  807 ARY(I,J)=BIC(I,J)+AID(I,J)
      DO 510 I=1,N
      RTD(I)=0.0
      RTDI(I)=0.0
      TTD(I)=0.0
      TTDI(I)=0.0
      DO 510 J=1,N
      RTD(I)=RTD(I)+RPI(I,J)*DR(J)
      RTDI(I)=RTDI(I)+RPI(I,J)*DI(J)
      TTD(I)=TTD(I)+TIPI(I,J)*DR(J)
      TTDI(I)=TTDI(I)+TIPI(I,J)*DI(J)
      RMI(I)=RTD(I)-TTDI(I)
  510 CPD(I)=RTDI(I)+TTD(I)
C     START OF FREE VIBRATION PROBLEM WITH INITIAL CONDITIONS GIVEN
   60 MCI=0
      DO 61 I=1,N
   61 AVAR(I)=-VAR(I)
      DO 100 I=1,N
      VMV(I)=0.0
      WMW(I)=0.0
      VMW(I)=0.0
      VCV(I)=0.0
      WCW(I)=0.0
      VCW(I)=0.0
      DO 71 J=1,N
      DO 71 K=1,N
      VMV(I)=VMV(I)+DISPR(K,I)*AMM(K,J)*DISPR(J,I)
      WMW(I)=WMW(I)+DISPI(K,I)*AMM(K,J)*DISPI(J,I)
```

```
      VMW(I)=VMW(I)+DISPR(K,I)*AMM(K,J)*DISPI(J,I)
      VCV(I)=VCV(I)+DISPR(K,I)*DM(K,J)*DISPR(J,I)
      WCW(I)=WCW(I)+DISPI(K,I)*DM(K,J)*DISPI(J,I)
   71 VCW(I)=VCW(I)+DISPR(K,I)*DM(K,J)*DISPI(J,I)
      ARN(I)=0.0
     0ARN(I)=ARN(I)-2.*AVAR(I)*(VMV(I)-WMW(I))-4.0*VAI(I)*VMW(I)+
     1VCV(I)-WCW(I)
      BRN(I)=0.0
     0BRN(I)=BRN(I)+2.0*VAI(I)*(VMV(I)-WMW(I))-4.*AVAR(I)*VMW(I)+
     12.0*VCW(I)
      DO 80 J=1,N
      DO 80 K=1,N
      GR(J,K)=0.0
      HR(J,K)=0.0
      G(J,K)=0.0
      H(J,K)=0.0
      B(J,K)=0.0
      HM(J,K)=0.0
      G(J,K)=G(J,K)+DISPR(J,I)*DISPR(K,I)-DISPI(J,I)*DISPI(K,I)
      H(J,K)=BRN(I)*G(J,K)
      G(J,K)=ARN(I)*G(J,K)
      B(J,K)=B(J,K)+DISPI(J,I)*DISPR(K,I)+DISPR(J,I)*DISPI(K,I)
      HM(J,K)=ARN(I)*B(J,K)
      B(J,K)=BRN(I)*B(J,K)
      GR(J,K)=(G(J,K)+B(J,K))/(ARN(I)**2+BRN(I)**2)
   80 HR(J,K)=(H(J,K)-HM(J,K))/(ARN(I)**2+BRN(I)**2)
      DO 81 J=1,N
      DO 81 K=1,N
      GRM(J,K)=0.0
      HRM(J,K)=0.0
      GRC(J,K)=0.0
      HRC(J,K)=0.0
      DO 81 L=1,N
      GRM(J,K)=GRM(J,K)+2.0*GR(J,L)*AMM(L,K)
      HRM(J,K)=HRM(J,K)+2.0*HR(J,L)*AMM(L,K)
      GRC(J,K)=GRC(J,K)+2.0*GR(J,L)*DM(L,K)
   81 HRC(J,K)=HRC(J,K)+2.0*HR(J,L)*DM(L,K)
      DO 82 J=1,N
      DO 82 K=1,N
      XOC(J,K)=0.0
      XOS(J,K)=0.0
      AGM(J,K)=AVAR(I)*GRM(J,K)
      BHM(J,K)=VAI(I)*HRM(J,K)
      BGM(J,K)=VAI(I)*GRM(J,K)
      AHM(J,K)=AVAR(I)*HRM(J,K)
      XOC(J,K)=-AGM(J,K)+BHM(J,K)+GRC(J,K)
   82 XOS(J,K)=-BGM(J,K)-AHM(J,K)+HRC(J,K)
      IF(MP1)999,201,101
  101 KC=1
      PRINT 83,VAI(I)
   83 FORMAT(39H0  THE COEFFICIENT MATRIX OF VEL X COS(F8.3,2HT))
      DO 84 J=1,N
   84 PRINT 85,(GRM(J,K),K=1,N)
   85 FORMAT(4E20.8)
      PRINT 86,VAI(I)
   86 FORMAT(35H0  THE COEFFICIENT MATRIX OF X COS(F8.3,2HT))
      DO 87 J=1,N
   87 PRINT 85,(XOC(J,K),K=1,N)
      PRINT 88,VAI(I)
   88 FORMAT(39H0  THE COEFFICIENT MATRIX OF VEL X SIN(F8.3,2HT))
      DO 89 J=1,N
   89 PRINT 85,(HRM(J,K),K=1,N)
```

```
      PRINT 90,VAI(I)
   90 FORMAT(35H0  THE COEFFICIENT MATRIX OF X SIN(F8.3,2HT))
      DO 91 J=1,N
   91 PRINT 85,(XOS(J,K),K=1,N)
      GO TO 100
  201 DO 202 J=1,N
      AB(J)=0.0
      CD(J)=0.0
      DO 202 K=1,N
      AB(J)=AB(J)+GRM(J,K)*VO(K)+XOC(J,K)*XO(K)
  202 CD(J)=CD(J)+HRM(J,K)*VO(K)+XOS(J,K)*XO(K)
      IF(MP2)999,301,203
  203 KC=2
      PRINT 204 VAR(I),VAI(I)
  204 FORMAT(33H0 THE COEFFICIENT COLUMN FOR EXP(F8.3,6HT)COS(F8.3,2HT))
  205 PRINT 206(AB(J),J=1,N)
  206 FORMAT(6E20.4)
      PRINT 207 VAR(I),VAI(I)
  207 FORMAT(33H0 THE COEFFICIENT COLUMN FOR EXP(F8.3,6HT)SIN(F8.3,2HT))
  208 PRINT 206(CD(J),J=1,N)
      GO TO 100
  301 IF(MP3)999,400,302
  302 KC=3
      DO 709 J=1,N
      XCO(J)=0.0
      VCO(J)=0.0
      XCO(J)=XO(J)-RMI(J)
  709 VCO(J)=VO(J)+WO*CPD(J)
      DO 710 J=1,N
      ABC(J)=0.0
      CDC(J)=0.0
      DO 710 K=1,N
      ABC(J)=ABC(J)+GRM(J,K)*VCO(K)+XOC(J,K)*XCO(K)
  710 CDC(J)=CDC(J)+HRM(J,K)*VCO(K)+XOS(J,K)*XCO(K)
      DO 711 J=1,N
      ABGR(J,I)=ABC(J)
  711 CDGR(J,I)=CDC(J)
  100 CONTINUE
      IF(KC-2)150,160,333
  150 MP1=0
      PRINT 102
  102 FORMAT(47H0 FREE VIBRATION OPTION NO. 1 HAS BEEN EXECUTED//)
      GO TO 60
  160 MP2=0
      PRINT 103
  103 FORMAT(47H0 FREE VIBRATION OPTION NO. 2 HAS BEEN EXECUTED//)
      GO TO 60
  333 DO 360 I=1,N
      IF(IC)999,717,712
  712  SI=0.0
      DO 713 K=1,NPTS
      Y(K)=0.0
      X(K)=ST
      DO 706 J=1,N
     0Y(K)=Y(K)+ABGR(I,J)*EXPF(VAR(J)*X(K))*COSF(VAI(J)*X(K))+CDGR(I,J)*
     1EXPF(VAR(J)*X(K))*SINF(VAI(J)*X(K))
  706 CONTINUE
  713 ST=ST+STEP
      IF(WO)999,718,717
  717 ST=0.0
      DO 716 K=1,NPTS
      X(K)=ST
```

```
      Y(K)=Y(K)+RPI(I)*COSF(WO*X(K))-CPD(I)*SINF(WO*X(K))
      ST=ST+STEP
  716 CONTINUE
  718 TEST(I)=0.0
      DO 715 J=1,NPTS
      IF(ABSF(Y(J))-TEST(I))715,715,714
  714 TEST(I)=ABSF(Y(J))
  715 CONTINUE
      WRITE TAPE 6,Y
  360 CONTINUE
      SCALE=0.0
      DO 721 I=1,N
      IF(TEST(I)-SCALE)721,721,722
  722 SCALE=TEST(I)
  721 CONTINUE
      YSCALE=SCALE/2.0
      REWIND 6
      DO 730 I=1,N
      MODCURV=0
      LABEL=LAB(I)
      DO 731 K=1,12
  731 ITITLE(K)=8H
      IF(WO)733,732,733
  732 ITITLE(1)=8H MIKLOS,
      ITITLE(2)=8H T.J.
      ITITLE(3)=8H     JOB
      ITITLE(4)=8H 0199
      ITITLE(5)=8H BOX M
      ITITLE(8)=8H FREE VI

      ITITLE(9)=8HBRATION
      GO TO 734
  733 IF(IC)999,735,736
  736 ITITLE(1)=8H MIKLOS,
      ITITLE(2)=8H T.J.
      ITITLE(3)=8H     JOB
      ITITLE(4)=8H BOX M
      ITITLE(7)=8H TRANSIE
      ITITLE(8)=8HNT PLUS
      ITITLE(9)=8HSTEADY S
      ITITLE(10)=8HTATE
      GO TO 734
  735 ITITLE(1)=8H MIKLOS,
      ITITLE(2)=8H T.J.
      ITITLE(3)=8H     JOB
      ITITLE(4)=8H BOX M
      ITITLE(7)=8H STEADY
      ITITLE(8)=8HSTATE VI
      ITITLE(9)=8HBRATION
  734 READ TAPE 6,Y
     0CALL DRAW(NPTS,X,Y,MODCURV,0,LABEL,ITITLE,0,YSCALE,2,0,2,0,8,4,1,
     1LAST)
  730 CONTINUE
  400 IF(MP4)999,500,401
  401 DO 450 I=1,N
      DO 410 J=1,N
      DO 410 K=1,N
      GR(J,K)=0.0
      HR(J,K)=0.0
      G(J,K)=0.0
      H(J,K)=0.0
      B(J,K)=0.0
```

```
      HM(J,K)=0.0
      G(J,K)=G(J,K)+DISPR(J,I)*DISPR(K,I)-DISPI(J,I)*DISPI(K,I)
      H(J,K)=HRN(I)*G(J,K)
      G(J,K)=ARN(I)*G(J,K)
      B(J,K)=B(J,K)+DISPI(J,I)*DISPR(K,I)+DISPR(J,I)*DISPI(K,I)
      HM(J,K)=ARN(I)*B(J,K)
      B(J,K)=BRN(I)*B(J,K)
      GR(J,K)=2.0*(G(J,K)+B(J,K))/(ARN(I)**2+BRN(I)**2)
410   HR(J,K)=2.0*(H(J,K)-HM(J,K))/(ARN(I)**2+BRN(I)**2)
      PRINT 411 I
4110  FORMAT(77H0 THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF
     1THE COS TERM FOR MODEI3,1X,2HIS)
      DO 412 J=1,N
412   PRINT 413 (GR(J,K),K=1,N)
413   FORMAT (6E20.8)
      PRINT 414 I
4140  FORMAT(77H0 THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF
     1THE SIN TERM FOR MODEI3,1X,2HIS)
      DO 415 J=1,N
415   PRINT 413(HR(J,K),K=1,N)
450   CONTINUE
      PRINT 104
104   FORMAT(49H0 FORCED VIBRATION OPTION NO. 4 HAS BEEN EXECUTED//)
500   IF(MP5)999,999,800
800   PRINT 801
801   FORMAT(30H0 THE REAL PART OF THE INVERSE)
      DO 802 I=1,N
802   PRINT 803 (RPI(I,J),J=1,N)
803   FORMAT(8E15.4)
      PRINT 804
804   FORMAT(30H0 THE IMAG PART OF THE INVERSE)
      DO 805 I=1,N
805   PRINT 803(IIPI(I,J),J=1,N)
      PRINT 808
808   FORMAT(38H0 THE REAL PART OF THE IDENTITY MATRIX)
      DO 809 I=1,N
809   PRINT 810(RIN(I,J),J=1,N)
810   FORMAT(8E15.4)
      PRINT 811
811   FORMAT(38H0 THE IMAG PART OF THE IDENTITY MATRIX)
      DO 812 I=1,N
812   PRINT 810(ARY(I,J),J=1,N)
511   PRINT 512 W0
512   FORMAT(32H0 THE COEFFICIENT COLUMN OF COS(F6.3,2HT))
      PRINT 513(RMI(I),I=1,N)
513   FORMAT(6E20.4)
      PRINT 514 W0
514   FORMAT(32H0 THE COEFFICIENT COLUMN OF SIN(F6.3,2HT))
      PRINT 513(CPD(I),I=1,N)
105   PRINT 106
106   FORMAT(49H0 FORCED VIBRATION OPTION NO. 5 HAS BEEN EXECUTED//)
      GO TO 999
941   PRINT 942
9420  FORMAT(90H0 SYSTEM CONTAINS TWO NATURAL FREQUENCIES OF EQUAL MAGNI
     1TUDE. PROGRAM HAS BEEN TERMINATED.)
      GO TO 1000
951   PRINT 952
9520  FORMAT(106H0 PROGRAM HAS BEEN TERMINATED DUE TO A NATURAL FREQUENC
     1Y OF ZERO MAGNITUDE. SYSTEM IS PROBABLY OVERDAMPED.)
999   GO TO 1000
998   END
      SUBROUTINE INVERT (A,N,D)
```

```
C     PROGRAM FOR FINDING THE INVERSE OF A NXN MATRIX
     0DIMENSION AM(10,10),DM(10,10),S(10,10),AMS(10,10),AMD(10,10),VALR
     1(20),VALI(20),VECR(20,20),VECI(20,20),AR(20,20),AI(20,20),BR(20,20
     2),BI(20,20),UR(20,20),UI(20,20),XR(20),XI(20),YR(20),YI(20),ZR(20)
     3,ZI(20)
     0COMMON AM,DM,S,AMS,AMD,VALR,VALI,VECR,VECI,AR,AI,BR,BI,UR,UI,XR,XI
     1,YR,YI,ZR,ZI
      DIMENSION A(10,10),L(10),M(10)
C     SEARCH FOR LARGEST ELEMENT
      D=1.0
      DO80 K=1,N
      L(K)=K
      M(K)=K
      BIGA=A(K,K)
      DO20 I=K,N
      DO20 J=K,N
      IF(ABSF(BIGA)-ABSF(A(I,J))) 10,20,20
   10 BIGA=A(I,J)
      L(K)=I
      M(K)=J
   20 CONTINUE
C     INTERCHANGE ROWS
      J=L(K)
      IF(L(K)-K) 35,35,25
   25 DO30 I=1,N
      HOLD=-A(K,I)
      A(K,I)=A(J,I)
   30 A(J,I)=HOLD
C     INTERCHANGE COLUMNS
   35 I=M(K)
      IF(M(K)-K) 45,45,37
   37 DO40 J=1,N
      HOLD=-A(J,K)
      A(J,K)=A(J,I)
   40 A(J,I)=HOLD
C     DIVIDE COLUMN BY MINUS PIVOT
   45 DO55 I=1,N
   46 IF(I-K)50,55,50
   50 A(I,K)=A(I,K)/(-A(K,K))
   55 CONTINUE
C     REDUCE MATRIX
      DO65 I=1,N
      DO65 J=1,N
   56 IF(I-K) 57,65,57
   57 IF(J-K) 60,65,60
   60 A(I,J)=A(I,K)*A(K,J)+A(I,J)
   65 CONTINUE
C     DIVIDE ROW BY PIVOT
      DO75 J=1,N
   68 IF(J-K)70,75,70
   70 A(K,J)=A(K,J)/A(K,K)
   75 CONTINUE
C     CONTINUED PRODUCT OF PIVOTS
      D=D*A(K,K)
C     REPLACE PIVOT BY RECIPROCAL
      A(K,K)=1.0/A(K,K)
   80 CONTINUE
C     FINAL ROW AND COLUMN INTERCHANGE
      K=N
  100 K=(K-1)
      IF(K) 150,150,103
  103 I=L(K)
```

```
      IF(I-K) 120,120,105
  105 DO110 J=1,N
      HOLD=A(J,K)
      A(J,K)=-A(J,I)
  110 A(J,I)=HOLD
  120 J=M(K)
      IF(J-K) 100,100,125
  125 DO130 I=1,N
      HOLD=A(K,I)
      A(K,I)=-A(J,I)
  130 A(J,I)=HOLD
      GO TO 100
  150 RETURN
      END
     0SUBROUTINE MATSUB (M,IEG,IVEC,ALRS,ALIS,GBR,GBI,IDET,MIT,MITS,EP1,
     1EP2)
     0DIMENSION AM(10,10),DM(10,10),S(10,10),AMS(10,10),AMD(10,10),VALR
     1(20),VALI(20),VECR(20,20),VECI(20,20),AR(20,20),AI(20,20),BR(20,20
     2),BI(20,20),CR(20,20),CI(20,20),XR(20),XI(20),YR(20),YI(20),ZR(20)
     3,ZI(20)
     0COMMON AM,DM,S,AMS,AMD,VALR,VALI,VECR,VECI,AR,AI,BR,BI,CR,CI,XR,XI
     1,YR,YI,ZR,ZI
      I NE=1
      I WO=2
      N=M
      SUMR=0.0
      SUMI=0.0
      PRDR=1.0
      PRDI=0.0
      TRACER=0.0
      TRACEI=0.0
      DO 450 I=1,N
      TRACER=TRACER+CR(I,I)
  450 TRACEI=TRACEI+CI(I,I)
C     SET UP MATRICES
      DO 519 I=1,N
      DO 519 J=1,N
      BR(I,J)=CR(I,J)
      AR(I,J)=CR(I,J)
      BI(I,J)=CI(I,J)
  519 AI(I,J)=CI(I,J)
C     EVALUATE DETERMINENT
      ASSIGN 520 TO IA
      ASSIGN 811 TO ID
      MM=M
      INTER=0
      GO TO 535
  520 DETR=1.0
      DETI=0.0
      DO 522 K=1,M
      T1=DETR*AR(K,K)-DETI*AI(K,K)
      DETI=DETR*AI(K,K)+DETI*AR(K,K)
  522 DETR=T1
      INTER=XMODF(INTER,2)
      IF (INTER) 1000,917,810
  810 DETR=-DETR
      DETI=-DETI
  917 GO TO ID
  811 PRINT 557,TRACER,TRACEI.DETR,DETI
  557 FORMAT (19H   TRACE OF MATRIX= 2E18.8,
     125H   DETERMINANT OF MATRIX= 2E18.8)
      ASSIGN 912 TO ID
```

```
      ASSIGN 530 TO IA
      ASSIGN 40 TO IB
      ASSIGN 523 TO IC
      ISL=-1
      GO TO 92
  523 ISL=0
C     EIGENVALUE GUESS OR ORIGIN TRANSLATION
    9 ALR=ALRS
      ALI=ALIS
      IT=1
C     EIGENVECTOR GUESS
  403 DO 504 I=1,N
      XR(I)=1.0
  504 XI(I)=0.0
    4 DO 5 I=1,N
      AR(I,I)=AR(I,I)-ALR
    5 AI(I,I)=AI(I,I)-ALI
C     FIRST ITERATION - POWER METHOD
      IJ=1
   10 BIG=0.
C     COMPUTE Y=(A-ALPHA)*X
      DO 13 I=1,N
      YR(I)=0.
      YI(I)=0.
      DO 11 J=1,N
      YR(I)=YR(I)+AR(I,J)*XR(J)-AI(I,J)*XI(J)
   11 YI(I)=YI(I)+AI(I,J)*XR(J)+AR(I,J)*XI(J)
      AM=YR(I)**2+YI(I)**2
      IF (AM-BIG) 13,13,12

   12 BIG=AM
      JJ=I
   13 CONTINUE
      IF(BIG) 109,106,109
C     EXACT EIGENVALUE AND EIGENVECTOR - Y=0. FLAG=1000
  106 ICT=1000
      DO 108 I=1,N
      JJ=I
      IF (XR(I)-1.0) 108,118,108
  118 ISL=1
      GO TO 92
  108 CONTINUE
      PRINT 650
  650 FORMAT (48H  ERROR. EIGENVECTOR NOT NORMALIZED IN METHOD 1.)



      GO TO 990
C     MU RAYLEIGH QUOTIENT - (Y,X)/(X,X)=MU
  109 RQNR=0.
      RQNI=0.
      RQD=0.
      DO 14 I=1,N
      RQNR=RQNR+XR(I)*YR(I)+XI(I)*YI(I)
      RQNI=RQNI+XR(I)*YI(I)-XI(I)*YR(I)
   14 RQD=RQD+XR(I)**2+XI(I)**2
      AMUR=RQNR/RQD
      AMUI=RQNI/RQD
      AMM=AMUR**2+AMUI**2
      IF (IEG) 1000,81,80
   80 ALRC=AMUR+ALR
      ALIC=AMUI+ALI
```

```
      PRINT 300,IONE,IJ,ALRC,ALIC
  300 FORMAT (2I4,2E20.8)
C     TEST FIRST ITERATION
C     MAGNITUDE OF (Y-MU*X)=TS
   81 TS=0.
      DO 15 I=1,N
   150TS=TS+(YR(I)-AMUR*XR(I)+AMUI*XI(I))**2+
     1(YI(I)-AMUR*XI(I)-AMUI*XR(I))**2
C     NORMALIZATION
      DO 16 I=1,N
      XR(I)=(YR(JJ)*YR(I)+YI(JJ)*YI(I))/BIG
   16 XI(I)=(YR(JJ)*YI(I)-YI(JJ)*YR(I))/BIG
      XR(JJ)=1.0
      XI(JJ)=0.0
  111 IF (TS/RQD-EP1) 20,20,18
   18 IF (IJ-MIT) 19,20,20
   19 IJ=IJ+1
      GO TO 10
C     SECOND ITERATION - INVERSE POWER METHOD
   20 ICT=IJ
      MIT2=MITS+IJ
      ALR=AMUR+ALR
      ALI=AMUI+ALI
      MM=N
      DO 310 I=1,N
      AR(I,I)=AR(I,I)-AMUR
  310 AI(I,I)=AI(I,I)-AMUI
      GO TO 29
   99 DO 100 I=1,N
      AR(I,I)=AR(I,I)-ALR
  100 AI(I,I)=AI(I,I)-ALI
   29 IJ=IJ+1
C     GAUSSIAN ELIMINATION - (A-ALPHA)*Y=X
  535 DO 27 I=2,MM
      IM1=I-1
      DO 27 J=1,IM1
   21 FM=AR(I,J)**2+AI(I,J)**2
      SM=AR(J,J)**2+AI(J,J)**2
      IF (FM-SM) 24,24,22
C     ROW INTERCHANGE - IF NECESSARY
   22 DO 23 K=J,MM
      T1=AR(J,K)
      T2=AI(J,K)
      AR(J,K)=AR(I,K)
      AI(J,K)=AI(I,K)
      AR(I,K)=T1
   23 AI(I,K)=T2
      T1=XR(J)
      T2=XI(J)
      XR(J)=XR(I)
      XI(J)=XI(I)
      XR(I)=T1
      XI(I)=T2
      T1=FM
      FM=SM
      SM=T1
      INTER=INTER+1
   24 IF (SM) 25,27,25
   25 IF (FM) 90,27,90
C     TRIANGULARIZATION
   90 RR=(AR(I,J)*AR(J,J)+AI(I,J)*AI(J,J))/SM
      RI=(AR(J,J)*AI(I,J)-AR(I,J)*AI(J,J))/SM
```

```
      DO 26 K=J,MM
      AR(I,K)=AR(I,K)-RR*AR(J,K)+RI*AI(J,K)
   26 AI(I,K)=AI(I,K)-RR*AI(J,K)-RI*AR(J,K)
      AR(I,J)=0.
      AI(I,J)=0.
      XR(I)=XR(I)-RR*XR(J)+RI*XI(J)
      XI(I)=XI(I)-RR*XI(J)-RI*XR(J)
   27 CONTINUE
      GO TO IA
  530 SMALL=1000.
      DO 28 K=1,MM
      IKK=K
      T1=AR(K,K)**2+AI(K,K)**2
      IF (T1) 750,752,750
  750 IF (T1-SMALL) 751,28,28
  751 SMALL=T1
      IZ=K
   28 CONTINUE
      GO TO IB
  752 IZ=IKK
      IF (ISL) 753,30,30
C     EXACT EIGENVALUE - (A-ALPHA) SINGULAR. FLAG=2000
   30 ISL=1
      ICI=2000
      DO 974 I=1,MM
      XR(I)=0.0
  974 XI(I)=0.0
  753 YR(IZ)=1.0
      YI(IZ)=0.0
      JJ=IZ
      BIG=1.0
      IF (IZ-MM) 33,32,33
   32 IZZ=2
      GO TO 95
   33 IZZ=IZ+1
      DO 31 I=IZZ,MM
      YR(I)=0.
   31 YI(I)=0.
      IZZ=MM-IZ+2
      IF (IZ-1) 95,49,95
C     BACKWARD SUBSTITUTION
   40 IZZ=1
   41 BIG=0.
   95 DO 46 I=IZZ,MM
      II=MM-I+1
      KK=II+1
      SR=0.
      SI=0.
      IF (I-1) 42,44,42
   42 DO 43 K=KK,MM
      SR=SR+AR(II,K)*YR(K)-AI(II,K)*YI(K)
   43 SI=SI+AR(II,K)*YI(K)+AI(II,K)*YR(K)
   44 T1=AR(II,II)**2+AI(II,II)**2
      YR(II)=(AR(II,II)*(XR(II)-SR)+AI(II,II)*(XI(II)-SI))/T1
      YI(II)=(AR(II,II)*(XI(II)-SI)-AI(II,II)*(XR(II)-SR))/T1
      AM=YR(II)**2+YI(II)**2
      IF (AM-BIG) 46,46,45
   45 JJ=II
      BIG=AM
   46 CONTINUE
C     NORMALIZATION - X=NORMALIZED Y
   49 DO 47 I=1,MM
```

```
      XR(I)=(YR(JJ)*YR(I)+YI(JJ)*YI(I))/BIG
   47 XI(I)=(YR(JJ)*YI(I)-YI(JJ)*YR(I))/BIG
      XR(JJ)=1.0
      XI(JJ)=0.0
   92 DO 601 I=1,N
      DO 601 J=1,N
      AR(I,J)=BR(I,J)
  601 AI(I,J)=BI(I,J)
  116 IF (ISL) 755,50,60
  755 GO TO IC
C     ALPHA RAYLEIGH QUOTIENT - (AX,X)/(X,X)=ALPHA
   50 ALR=0.
      ALI=0.
      SUM=0.0
   55 DO 52 I=1,N
      YR(I)=0.
      YI(I)=0.
      DO 51 K=1,N
      YR(I)=YR(I)+AR(I,K)*XR(K)-AI(I,K)*XI(K)
   51 YI(I)=YI(I)+AR(I,K)*XI(K)+AI(I,K)*XR(K)
      ALR=ALR+XR(I)*YR(I)+XI(I)*YI(I)
      ALI=ALI+XR(I)*YI(I)-XI(I)*YR(I)
   52 SUM=SUM+XR(I)**2+XI(I)**2
      ALR=ALR/SUM
      ALI=ALI/SUM
      AM=ALR**2+ALI**2
      IF (IEG) 1000,83,82
   82 PRINT 300,ITWO,IF,ALR,ALI
C     TEST SECOND ITERATION
   83 TS=0.
      DO 53 I=1,N
      T1=YR(I)-ALR*XR(I)+ALI*XI(I)
      T2=YI(I)-ALR*XI(I)-ALI*XR(I)
   53 TS=TS+T1**2+T2**2
   93 IF (TS/SUM-EP2)60,60,301
  301 IF (IJ-MIT2) 99,400,400
  400 PRINT 401,IT
  401 FORMAT (54H      INVERSE POWER METHOD NOT CONVERGED ON TRY NUMBER
     1I5)
      IF (IT-3) 402,990,402
  402 ALR=ALR+GBR
      ALI=ALI+GBI
      IT=IT+1
      PRINT 820,ALR,ALI
      GO TO 4
   60 ISL=0
   63 PRINT 64, N,ALR,ALI,ICT,IJ
      VALR(N)=ALR
      VALI(N)=ALI
   64 FORMAT (I5,15H TH EIGENVALUE= 2E18.8,53X,2I5)
      SUMR=SUMR+ALR
      SUMI=SUMI+ALI
      T1=PRDR*ALR-PRDI*ALI
      PRDI=PRDR*ALI+PRDI*ALR
      PRDR=T1
C     DEFLATION OF MATRIX
      IF (JJ-N) 61,65,61
C     PERMUTATION OPERATION
   61 T1=XR(JJ)
      T2=XI(JJ)
      XR(JJ)=XR(N)
      XI(JJ)=XI(N)
```

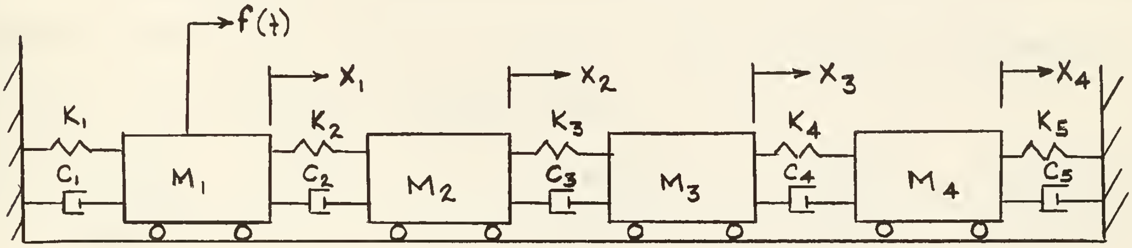
```
      XR(N)=T1
      XI(N)=T2
      DO 68 K=1,N
      T1=AR(JJ,K)
      T2=AI(JJ,K)
      AR(JJ,K)=AR(N,K)
      AI(JJ,K)=AI(N,K)
      AR(N,K)=T1
   68 AI(N,K)=T2
      DO 62 K=1,N
      T1=AR(K,JJ)
      T2=AI(K,JJ)
      AR(K,JJ)=AR(K,N)
      AI(K,JJ)=AI(K,N)
      AR(K,N)=T1
   62 AI(K,N)=T2
C     DEFLATION
   65 NN=N
      N=N-1
      DO 66 I=1,N
      DO 66 J=1,N
      AR(I,J)=AR(I,J)-XR(I)*AR(N+1,J)+XI(I)*AI(N+1,J)
   66 AI(I,J)=AI(I,J)-XR(I)*AI(N+1,J)-XI(I)*AR(N+1,J)
      DO 600 I=1,N
      DO 600 J=1,N
      BR(I,J)=AR(I,J)
  600 BI(I,J)=AI(I,J)
C     COMPUTE EIGENVECTOR AND/OR DETERMINANT AS REQUIRED
  910 IF (IDET) 1000,527,700
  527 IF (IVEC) 1000,525,700
  700 DO 702 I=1,M
      DO 702 J=1,M
      AR(I,J)=CR(I,J)
      AI(I,J)=CI(I,J)
      IF (I-J) 702,701,702
  701 AR(I,I)=AR(I,I)-ALR
      AI(I,I)=AI(I,I)-ALI
  702 CONTINUE
      MM=M
      INTER=0
      ASSIGN 911 TO IA
      GO TO 535
  911 ASSIGN 530 TO IA
      IF (IDET) 1000,914,520
  912 PRINT 913,DETR,DETI
  913 FORMAT (58X,12HDETERMINANT= 2E18.8)
      ZLAG=SQRTF(AR(1,1)**2+AI(1,1)**2)
      ZLIT=ZLAG
      DO 923 I=2,M
      ZMAGT=SQRTF(AR(I,I)**2+AI(I,I)**2)
      IF (ZLAG-ZMAGT) 922,920,920
  920 IF (ZLIT - ZMAGT) 923,923,921
  921 ZLIT=ZMAGT
      GO TO 923
  922 ZLAG=ZMAGT
  923 CONTINUE
      PRINT 924,ZLAG,ZLIT
  9240FORMAT (70H  LARGEST AND SMALLEST MAGNITUDES OF DIAGONAL ELEMENTS
     1OF TRI. MATRIX= 2E18.8/)
  914 ISL=-1
      IF (IVEC) 1000,916,915
  915 DO 703 I=1,M
```

```
      XR(I)=0.
  703 XI(I)=0.
      ASSIGN 753 TO IB
      ASSIGN 704 TO IC
      GO TO 530
  910 ASSIGN 525 TO IC
      GO TO 92
  704 PRINT 705, (XR(I),XI(I),I=1,M)
      DO 707 I=1,M
      VECR(I,NN)=XR(I)
  707 VECI(I,NN)=XI(I)
  705 FORMAT (30H      ASSOCIATED EIGENVECTOR IS/(2E20.8))
      ASSIGN 40 TO IB
  525 IF (N-1) 526,67,523
   67 ALR=AR(1,1)
      ALI=AI(1,1)
      SUMR=SUMR+ALR
      SUMI=SUMI+ALI
      T1=PRDR*ALR-PRDI*ALI
      PRDI=PRDR*ALI+PRDI*ALR
      PRDR=T1
      PRINT 320,ALR,ALI
      VALR(1)=ALR
      VALI(1)=ALI
  320 FORMAT (20H   FINAL EIGENVALUE= 2E18.8)
      N=0
      NN=1
      GO TO 910
 1000 STOP
  820 FORMAT(5X6HALPHA= 2E20.8)
  526 PRINT 321,SUMR,SUMI,PRDR,PRDI
  3210FORMAT (21H  SUM OF EIGENVALUES= 2E18.8,
     125H  PRODUCT OF EIGENVALUES= 2E18.8//)
  990 CONTINUE
      END
      END
```

APPENDIX E

SAMPLE PROBLEM

The sample problem presented is that of a four degree of freedom system shown in the figure below. In order to show a comparison of data for damped and undamped systems, the example chosen is one in which the natural frequencies of the undamped system may be obtained with relative ease. The program is not restricted to problems with the symmetry displayed in the example.



Let $M_1 = M_2 = M_3 = M_4 = 10 \text{ lb sec}^2/\text{in}$

$K_1 = K_2 = K_3 = K_4 = 1000 \text{ lb/in}$

$C_1 = C_2 = C_3 = C_4 = 30 \text{ lb sec/in}$

For the undamped case the frequency determinate then becomes

$$\begin{vmatrix} 2K - M\omega^2 & -K & 0 & 0 \\ -K & 2K - M\omega^2 & -K & 0 \\ 0 & -K & 2K - M\omega^2 & -K \\ 0 & 0 & -K & 2K - M\omega^2 \end{vmatrix}$$

Expanding the determinate

$$\omega^8 - 8\left(\frac{K}{M}\right)\omega^6 + 21\left(\frac{K}{M}\right)^2\omega^4 - 20\left(\frac{K}{M}\right)^3\omega^2 + 5\left(\frac{K}{M}\right)^4 = 0$$

The roots of the frequency equation are:

$\omega_1 = 6.18034, \omega_2 = 11.75571, \omega_3 = 16.18034, \omega_4 = 19.02113$

The problem was programmed with a forcing function f(t) = 100sin (60t), and an initial displacement on M_4 of 0.5 inches. A step size of 0.005 seconds was used for the graphical option. All five options were called for and the resulting output is shown on the following pages.

In addition to the output presented for C = 30 lbsec/in runs were made with C = 0.01, 0.10, and 1.0 lb sec/in. The frequencies and damping ratios for all runs are tabulated below.

C lb sec/in	ξ_1	ω_1, rad/sec	ξ_2	ω_2, rad/sec
0.01	—	6.18034	0.0001	11.75571
0.10	0.0003	6.18034	0.0006	11.75570
1.00	0.0031	6.18028	0.0067	11.75530
30.00	0.0098	6.12699	0.1821	11.38430

C lb sec/in	ξ_3	ω_3, rad/sec	ξ_4	ω_4, rad/sec
0.01	0.0001	16.18034	0.0001	19.02113
0.10	0.0008	16.18033	0.0009	19.02111
1.00	0.0081	16.17928	0.0095	19.01941
30.00	0.2584	15.19737	0.3118	17.40396

```
         MATRIX ANALYSIS OF A MULTI-DEGREE OF FREEDOM VIBRATION SYSTEM WITH VISCOUS DAMPING.

   SYSTEM OF ORDER    4

   INERTIA MATRIX
         .100E+02        .000E+00        .000E+00        .000E+00
         .000E+00        .100E+02        .000E+00        .000E+00
         .000E+00        .000E+00        .100E+02        .000E+00
         .000E+00        .000E+00        .000E+00        .100E+02

   DAMPING MATRIX
         .600E+02       -.300E+02        .000E+00        .000E+00
        -.300E+02        .600E+02       -.300E+02        .000E+00
         .000E+00       -.300E+02        .600E+02       -.300E+02
         .000E+00        .000E+00       -.300E+02        .600E+02

   STIFFNESS MATRIX
         .200E+04       -.100E+04        .000E+00        .000E+00
        -.100E+04        .200E+04       -.100E+04        .000E+00
         .000E+00       -.100E+04        .200E+04       -.100E+04
         .000E+00        .000E+00       -.100E+04        .200E+04

THE INITIAL DISPLACEMENTS ARE

         .000E+00        .000E+00        .000E+00        .500E+00

THE INITIAL CONDITIONS OF VELOCITY ARE

         .000E+00        .000E+00        .000E+00        .000E+00

THE REAL PART OF THE AMPLITUDE OF THE DRIVING FORCE

         .100E+03        .000E+00        .000E+00        .000E+00

THE IMAG PART OF THE AMPLITUDE OF THE DRIVING FORCE

         .000E+00        .000E+00        .000E+00        .000E+00

THE FREQUENCY OF THE FORCING FUNCTION IS    60.0000RADIANS PER SECOND

THE STEP SIZE FOR THE GRAPHICAL OPTION IS .0050SECONDS

      END OF INPUT DATA
```

```
TRACE OF MATRIX=     -.24000000E+02      .00000000E+00   DETERMINANT OF MATRIX=      .50000000E+09      .00000000E+00
 8 TH EIGENVALUE=     -.39270510E+01     -.15696550E+02                                                   20    27
  ASSOCIATED EIGENVECTOR IS
   -.15000000E-01       .59955484E-01
    .92705098E-02      -.37054527E-01
    .92705098E-02      -.37054527E-01
   -.15000000E-01       .59955484E-01
    .10000000E+01       .00000000E+00
   -.61803399E+00      -.12372172E-10
   -.61803399E+00       .54640296E-11
    .10000000E+01       .95539312E-11
 7 TH EIGENVALUE=     -.54270510E+01     -.18230483E+02                                                   20    27
  ASSOCIATED EIGENVECTOR IS
   -.92705098E-02       .31141383E-01
    .15000000E-01      -.50387816E-01
   -.15000000E-01       .50387816E-01
    .92705098E-02      -.31141383E-01
    .61803399E+00      -.10998085E-09
   -.10000000E+01       .10173985E-09
    .10000000E+01       .00000000E+00
   -.61803399E+00      -.26054705E-10
 6 TH EIGENVALUE=     -.54270510E+01      .18230483E+02                                                   20    23
  ASSOCIATED EIGENVECTOR IS
   -.92705098E-02      -.31141383E-01
    .15000000E-01       .50387816E-01
   -.15000000E-01      -.50387816E-01
    .92705098E-02       .31141383E-01
    .61803399E+00       .61674151E-09
   -.10000000E+01      -.44087270E-09
    .10000000E+01       .00000000E+00
   -.61803399E+00       .11290372E-09
 5 TH EIGENVALUE=     -.39270510E+01      .15696550E+02                                                   20    22
  ASSOCIATED EIGENVECTOR IS
   -.15000000E-01      -.59955484E-01
    .92705098E-02       .37054527E-01
    .92705098E-02       .37054527E-01
   -.15000000E-01      -.59955484E-01
    .10000000E+01      -.19107862E-10
   -.61803399E+00       .71142697E-10
   -.61803399E+00      -.39790393E-10
    .10000000E+01       .00000000E+00
 4 TH EIGENVALUE=     -.20729490E+01     -.11571494E+02                                                   20    26
  ASSOCIATED EIGENVECTOR IS
   -.15000000E-01       .83732120E-01
   -.92705095E-02       .51749296E-01
    .92705098E-02      -.51749296E-01
    .15000000E-01      -.83732120E-01
    .10000000E+01       .00000000E+00
    .61803399E+00      -.43108852E-08
   -.61803399E+00       .10941793E-08
   -.10000000E+01       .40973763E-08
 3 TH EIGENVALUE=     -.20729490E+01      .11571494E+02                                                   18    20
  ASSOCIATED EIGENVECTOR IS
   -.15000000E-01      -.83732120E-01
   -.92705100E-02      -.51749296E-01
    .92705099E-02       .51749296E-01
    .15000000E-01       .83732120E-01
    .10000000E+01       .00000000E+00
    .61803399E+00      -.30441634E-08

   -.61803399E+00       .75569795E-09
   -.10000000E+01       .29058798E-08
```

ITERATIONS REQ'D FOR CONVERGENCE - POWER METHOD

TOTAL ITERATIONS REQ'D FOR CONVERGENCE

```
2 TH EIGENVALUE=    -.57294902E+00    -.61537249E+01                                                             20   25
  ASSOCIATED EIGENVECTOR IS
   -.92705098E-02     .99569361E-01
   -.15000000E-01     .16110661E+00
   -.15000000E-01     .16110661E+00
   -.92705098E-02     .99569361E-01
    .61803399E+00     .16051184E-10
    .10000000E+01     .00000000E+00
    .10000000E+01    -.64233551E-10
    .61803399E+00    -.16258136E-09
 FINAL EIGENVALUE=   -.57294902E+00     .61537249E+01
  ASSOCIATED EIGENVECTOR IS
   -.92705098E-02    -.99569361E-01
   -.15000000E-01    -.16110661E+00
   -.15000000E-01    -.16110661E+00
   -.92705098E-02    -.99569361E-01
    .61803399E+00    -.21955839E-10
    .10000000E+01     .00000000E+00
    .10000000E+01     .78542994E-10
    .61803399E+00     .18759387E-09
SUM OF EIGENVALUES=  -.24000000E+02     .58207661E-09  PRODUCT OF EIGENVALUES=    .50000000E+09   -.45410156E+00



 THE NATURAL FREQUENCIES ARE
    .61269944E+01     .11384303E+02     .15197366E+02     .17403955E+02



  MODAL MATRIX, REAL PART
    .61803399E+00     .10000000E+01     .10000000E+01     .61803399E+00
    .10000000E+01     .61803399E+00    -.61803399E+00    -.10000000E+01
    .10000000E+01    -.61803399E+00    -.61803399E+00     .10000000E+01
    .61803399E+00    -.10000000E+01     .10000000E+01    -.61803399E+00


  MODAL MATRIX, IMAG PART
   -.21955839E-10     .00000000E+00    -.19107862E-10     .61674151E-09
    .00000000E+00    -.30441634E-08     .71142697E-10    -.44087270E-09
    .78542994E-10     .75569795E-09    -.39790393E-10     .00000000E+00
    .18759387E-09     .29058798E-08     .00000000E+00     .11290372E-09


 THE COEFFICIENT MATRIX OF VEL X COS(   6.154T)
   -.43624026E-11    -.57676395E-11    -.29136369E-11     .32519677E-11
   -.57676395E-11    -.72435552E-11    -.26256819E-11     .65526704E-11
   -.29136369E-11    -.26256819E-11     .19921914E-11     .94066730E-11
    .32519677E-11     .65526704E-11     .94066730E-11     .10866338E-10


 THE COEFFICIENT MATRIX OF X COS(   6.154T)
    .13819660E+00     .22360680E+00     .22360680E+00     .13819660E+00
    .22360680E+00     .36180340E+00     .36180340E+00     .22360680E+00
    .22360680E+00     .36180340E+00     .36180340E+00     .22360680E+00
    .13819660E+00     .22360680E+00     .22360680E+00     .13819660E+00
```

THE COEFFICIENT MATRIX OF VEL X SIN(6.154T)

.22457390E-01	.36336820E-01	.36336820E-01	.22457390E-01
.36336820E-01	.58794210E-01	.58794210E-01	.36336820E-01
.36336820E-01	.58794210E-01	.58794210E-01	.36336820E-01
.22457390E-01	.36336820E-01	.36336820E-01	.22457390E-01

THE COEFFICIENT MATRIX OF X SIN(6.154T)

.12866940E-01	.20819146E-01	.20819146E-01	.12866939E-01
.20819146E-01	.33686085E-01	.33686085E-01	.20819145E-01
.20819146E-01	.33686085E-01	.33686085E-01	.20819145E-01
.12866940E-01	.20819145E-01	.20819145E-01	.12866939E-01

THE COEFFICIENT MATRIX OF VEL X COS(11.571T)

.13474983E-09	-.11901221E-10	-.59651733E-10	-.43892319E-10
-.11901221E-10	-.66180573E-10	.21958416E-10	.68054252E-10
-.59651733E-10	.21958416E-10	.22263741E-10	.34987015E-11
-.43892319E-10	.68054252E-10	.34987015E-11	-.46965194E-10

THE COEFFICIENT MATRIX OF X COS(11.571T)

.36180340E+00	.22360680E+00	-.22360680E+00	-.36180340E+00
.22360680E+00	.13819660E+00	-.13819660E+00	-.22360680E+00
-.22360680E+00	-.13819660E+00	.13819660E+00	.22360680E+00
-.36180340E+00	-.22360680E+00	.22360680E+00	.36180340E+00

THE COEFFICIENT MATRIX OF VEL X SIN(11.571T)

.31266783E-01	.19323934E-01	-.19323934E-01	-.31266782E-01
.19323934E-01	.11942848E-01	-.11942848E-01	-.19323934E-01
-.19323934E-01	-.11942848E-01	.11942848E-01	.19323934E-01
-.31266782E-01	-.19323934E-01	.19323934E-01	.31266782E-01

THE COEFFICIENT MATRIX OF X SIN(11.571T)

.64814445E-01	.40057530E-01	-.40057530E-01	-.64814446E-01
.40057531E-01	.24756916E-01	-.24756916E-01	-.40057531E-01
-.40057530E-01	-.24756915E-01	.24756915E-01	.40057531E-01
-.64814446E-01	-.40057531E-01	.40057531E-01	.64814446E-01

THE COEFFICIENT MATRIX OF VEL X COS(15.697T)

-.38605907E-12	.16062244E-11	-.95076869E-12	.54374643E-13
.16062244E-11	-.18379411E-11	-.25763246E-12	.13340214E-11
-.95076869E-12	-.25763246E-12	.13226762E-11	-.12229717E-11
.54374643E-13	.13340214E-11	-.12229717E-11	.49480835E-12

THE COEFFICIENT MATRIX OF X COS(15.697T)

.36180340E+00	-.22360680E+00	-.22360680E+00	.36180340E+00
-.22360680E+00	.13819660E+00	.13819660E+00	-.22360680E+00
-.22360680E+00	.13819660E+00	.13819660E+00	-.22360680E+00
.36180340E+00	-.22360680E+00	-.22360680E+00	.36180340E+00

THE COEFFICIENT MATRIX OF VEL X SIN(15.697T)

.23049868E-01	-.14245602E-01	-.14245602E-01	.23049868E-01
-.14245602E-01	.88042662E-02	.88042662E-02	-.14245602E-01
-.14245602E-01	.88042662E-02	.88042662E-02	-.14245602E-01
.23049868E-01	-.14245602E-01	-.14245602E-01	.23049868E-01

THE COEFFICIENT MATRIX OF X SIN(15.697T)

.90518007E-01	-.55943205E-01	-.55943205E-01	.90518007E-01
-.55943205E-01	.34574802E-01	.34574802E-01	-.55943205E-01
-.55943205E-01	.34574802E-01	.34574802E-01	-.55943205E-01
.90518007E-01	-.55943205E-01	-.55943205E-01	.90518007E-01

THE COEFFICIENT MATRIX OF VEL X COS(18.230T)

.10813906E-10	-.10664918E-10	.52573745E-11	-.18644106E-11
-.10664918E-10	.62012254E-11	.25483639E-11	-.38156704E-11
.52573745E-11	.25483639E-11	-.11297953E-10	.92232140E-11
-.18644106E-11	-.38156704E-11	.92232140E-11	-.70850853E-11

THE COEFFICIENT MATRIX OF X COS(18.230T)

.13819660E+00	-.22360680E+00	.22360680E+00	-.13819660E+00
-.22360680E+00	.36180340E+00	-.36180340E+00	.22360680E+00
.22360680E+00	-.36180340E+00	.36180340E+00	-.22360680E+00
-.13819660E+00	.22360680E+00	-.22360680E+00	.13819660E+00

THE COEFFICIENT MATRIX OF VEL X SIN(18.230T)

.75805232E-02	-.12265544E-01	.12265544E-01	-.75805232E-02
-.12265544E-01	.19846067E-01	-.19846067E-01	.12265544E-01
.12265544E-01	-.19846067E-01	.19846067E-01	-.12265544E-01
-.75805232E-02	.12265544E-01	-.12265544E-01	.75805232E-02

THE COEFFICIENT MATRIX OF X SIN(18.230T)

.41139886E-01	-.66565734E-01	.66565734E-01	-.41139886E-01
-.66565734E-01	.10770562E+00	-.10770562E+00	.66565734E-01
.66565734E-01	-.10770562E+00	.10770562E+00	-.66565734E-01
-.41139886E-01	.66565734E-01	-.66565734E-01	.41139886E-01

FREE VIBRATION OPTION NO. 1 HAS BEEN EXECUTED

```
THE COEFFICIENT COLUMN FOR EXP(   -.573T)COS(   6.154T)
       .6910E-01          .1118E+00          .1118E+00          .6910E-01

THE COEFFICIENT COLUMN FOR EXP(   -.573T)SIN(   6.154T)
       .6433E-02          .1041E-01          .1041E-01          .6433E-02

THE COEFFICIENT COLUMN FOR EXP(  -2.073T)COS(  11.571T)
      -.1809E+00         -.1118E+00          .1118E+00          .1809E+00

THE COEFFICIENT COLUMN FOR EXP(  -2.073T)SIN(  11.571T)
      -.3241E-01         -.2003E-01          .2003E-01          .3241E-01

THE COEFFICIENT COLUMN FOR EXP(  -3.927T)COS(  15.697T)
       .1809E+00         -.1118E+00         -.1118E+00          .1809E+00

THE COEFFICIENT COLUMN FOR EXP(  -3.927T)SIN(  15.697T)
       .4526E-01         -.2797E-01         -.2797E-01          .4526E-01

THE COEFFICIENT COLUMN FOR EXP(  -5.427T)COS(  18.230T)

      -.6910E-01          .1118E+00         -.1118E+00          .6910E-01

THE COEFFICIENT COLUMN FOR EXP(  -5.427T)SIN(  18.230T)
      -.2057E-01          .3328E-01         -.3328E-01          .2057E-01

FREE VIBRATION OPTION NO. 2 HAS BEEN EXECUTED


GRAPH TITLED . .   MIKLOS, T.J.        JOB BOX M
                   TRANSIENT PLUS STEADY STATE                   . . HAS BEEN PLOTTED.


GRAPH TITLED . .   MIKLOS, T.J.        JOB BOX M
                   TRANSIENT PLUS STEADY STATE                   . . HAS BEEN PLOTTED.


GRAPH TITLED . .   MIKLOS, T.J.        JOB BOX M
                   TRANSIENT PLUS STEADY STATE                   . . HAS BEEN PLOTTED.


GRAPH TITLED . .   MIKLOS, T.J.        JOB BOX M
                   TRANSIENT PLUS STEADY STATE                   . . HAS BEEN PLOTTED.
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THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF THE COS TERM FOR MODE  1 IS
    -.43624026E-12      -.57676395E-12      -.29136369E-12       .32519677E-12
    -.57676395E-12      -.72435552E-12      -.26256819E-12       .65526704E-12
    -.29136369E-12      -.26256819E-12       .19921914E-12       .94066730E-12
     .32519677E-12       .65526704E-12       .94066730E-12       .10866338E-11

THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF THE SIN TERM FOR MODE  1 IS
     .22457390E-02       .36336820E-02       .36336820E-02       .22457390E-02
     .36336820E-02       .58794210E-02       .58794210E-02       .36336820E-02
     .36336820E-02       .58794210E-02       .58794210E-02       .36336820E-02
     .22457390E-02       .36336820E-02       .36336820E-02       .22457390E-02

THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF THE COS TERM FOR MODE  2 IS
     .13474983E-10      -.11901221E-11      -.59651733E-11      -.43892319E-11
    -.11901221E-11      -.66180573E-11       .21958416E-11       .68054252E-11
    -.59651733E-11       .21958416E-11       .22263741E-11       .34987015E-12
    -.43892319E-11       .68054252E-11       .34987015E-12      -.46965194E-11

THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF THE SIN TERM FOR MODE  2 IS
     .31266783E-02       .19323934E-02      -.19323934E-02      -.31266782E-02
     .19323934E-02       .11942848E-02      -.11942848E-02      -.19323934E-02
    -.19323934E-02      -.11942848E-02       .11942848E-02       .19323934E-02
    -.31266782E-02      -.19323934E-02       .19323934E-02       .31266782E-02


THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF THE COS TERM FOR MODE  3 IS
    -.38605907E-13       .16062244E-12      -.95076869E-13       .54374643E-14
     .16062244E-12      -.18379411E-12      -.25763246E-13       .13340214E-12
    -.95076869E-13      -.25763246E-13       .13226762E-12      -.12229717E-12
     .54374643E-14       .13340214E-12      -.12229717E-12       .49480835E-13

THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF THE SIN TERM FOR MODE  3 IS
     .23049868E-02      -.14245602E-02      -.14245602E-02       .23049868E-02
    -.14245602E-02       .88042662E-03       .88042662E-03      -.14245602E-02
    -.14245602E-02       .88042662E-03       .88042662E-03      -.14245602E-02
     .23049868E-02      -.14245602E-02      -.14245602E-02       .23049868E-02

THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF THE COS TERM FOR MODE  4 IS
     .10813906E-11      -.10664918E-11       .52573745E-12      -.18644106E-12
    -.10664918E-11       .62012254E-12       .25483639E-12      -.38156704E-12
     .52573745E-12       .25483639E-12      -.11297953E-11       .92232140E-12
    -.18644106E-12      -.38156704E-12       .92232140E-12      -.70850853E-12

THE COEFFICIENT MATRIX OF THE CONVOLUTION INTEGRAL OF THE SIN TERM FOR MODE  4 IS
     .75805232E-03      -.12265544E-02       .12265544E-02      -.75805232E-03
    -.12265544E-02       .19846067E-02      -.19846067E-02       .12265544E-02
     .12265544E-02      -.19846067E-02       .19846067E-02      -.12265544E-02
    -.75805232E-03       .12265544E-02      -.12265544E-02       .75805232E-03

FORCED VIBRATION OPTION NO. 4 HAS BEEN EXECUTED
```

THE REAL PART OF THE INVERSE

-.2900E-04	.5028E-06	.8121E-07	-.5627E-08
.5028E-06	-.2892E-04	.4971E-06	.8121E-07
.8121E-07	.4971E-06	-.2892E-04	.5028E-06
-.5627E-08	.8121E-07	.5028E-06	-.2900E-04

THE IMAG PART OF THE INVERSE

-.3147E-05	.1679E-05	-.6702E-07	-.2924E-08
.1679E-05	-.3214E-05	.1676E-05	-.6702E-07
-.6702E-07	.1676E-05	-.3214E-05	.1679E-05
-.2924E-08	-.6702E-07	.1679E-05	-.3147E-05

THE REAL PART OF THE IDENTITY MATRIX

.1000E+01	.9095E-12	.0000E+00	-.7105E-14
.9095E-12	.1000E+01	.9095E-12	.1137E-12
.1137E-12	.6821E-12	.1000E+01	.6821E-12
.1776E-14	-.5684E-13	.2274E-12	.1000E+01

THE IMAG PART OF THE IDENTITY MATRIX

-.1819E-11	.0000E+00	.5684E-13	.0000E+00
-.1819E-11	.1819E-11	-.1819E-11	.1137E-12
-.9948E-13	.2728E-11	-.1819E-11	.9095E-12
.0000E+00	-.1421E-13	.0000E+00	.0000E+00

THE COEFFICIENT COLUMN OF COS(60.000T)

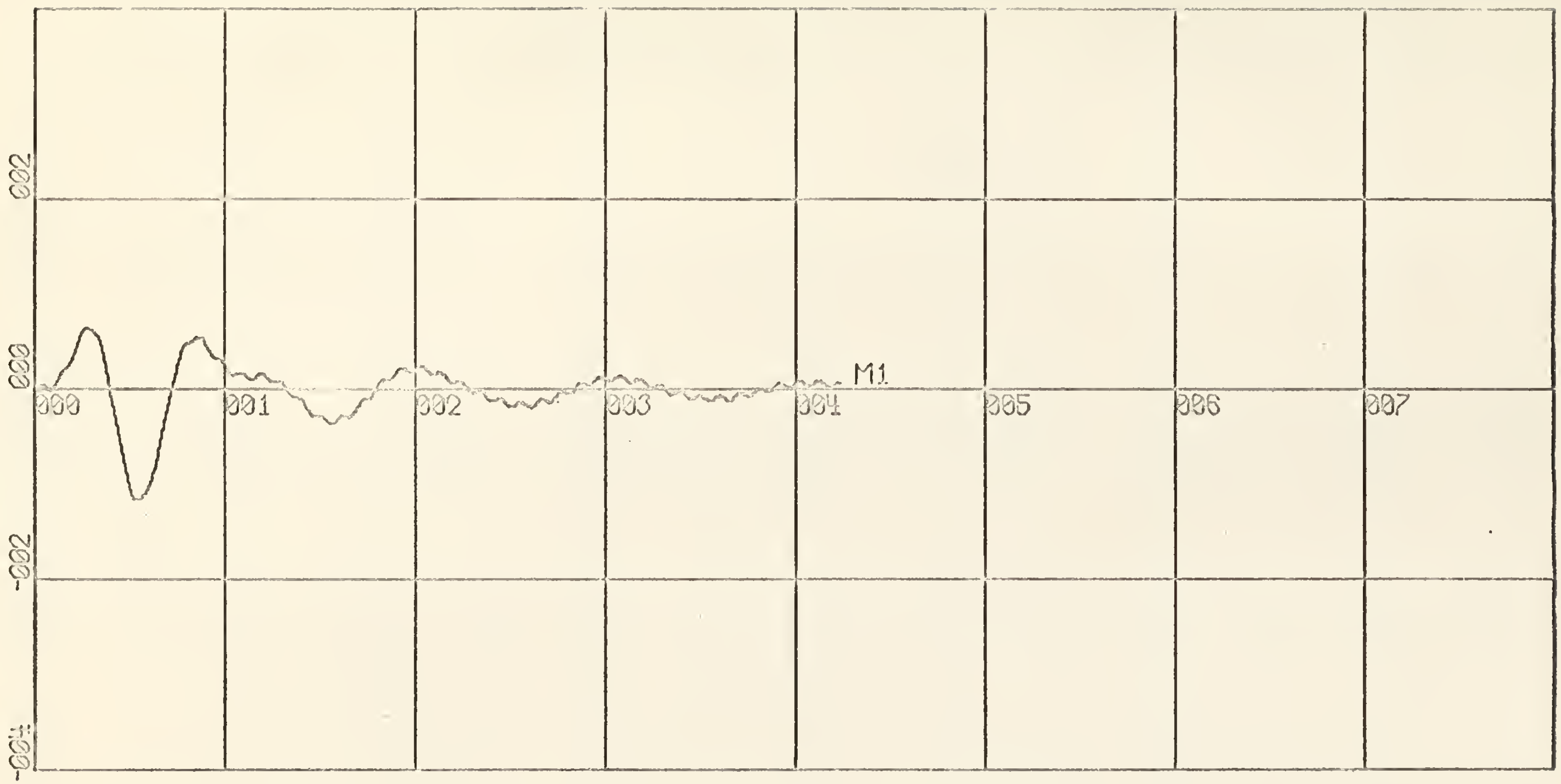
-.2900E-02	.5028E-04	.8121E-05	-.5627E-06

THE COEFFICIENT COLUMN OF SIN(60.000T)

-.3147E-03	.1679E-03	-.6702E-05	-.2924E-06

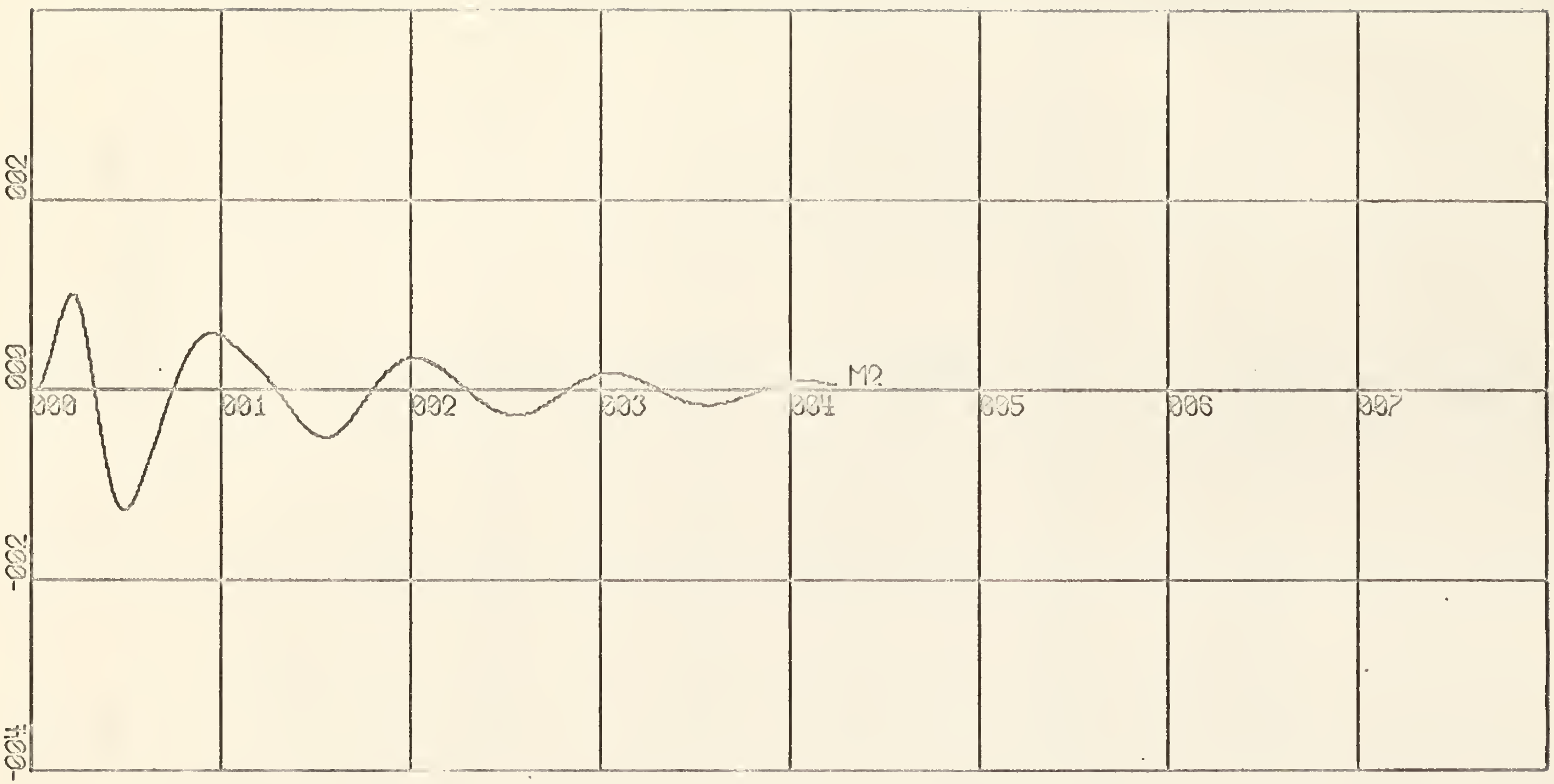
FORCED VIBRATION OPTION NO. 5 HAS BEEN EXECUTED

TIME, 6 MINUTES AND 55 SECONDS



X-SCALE = 1.00E+00 UNITS/INCH.
Y-SCALE = 2.00E-01 UNITS/INCH.

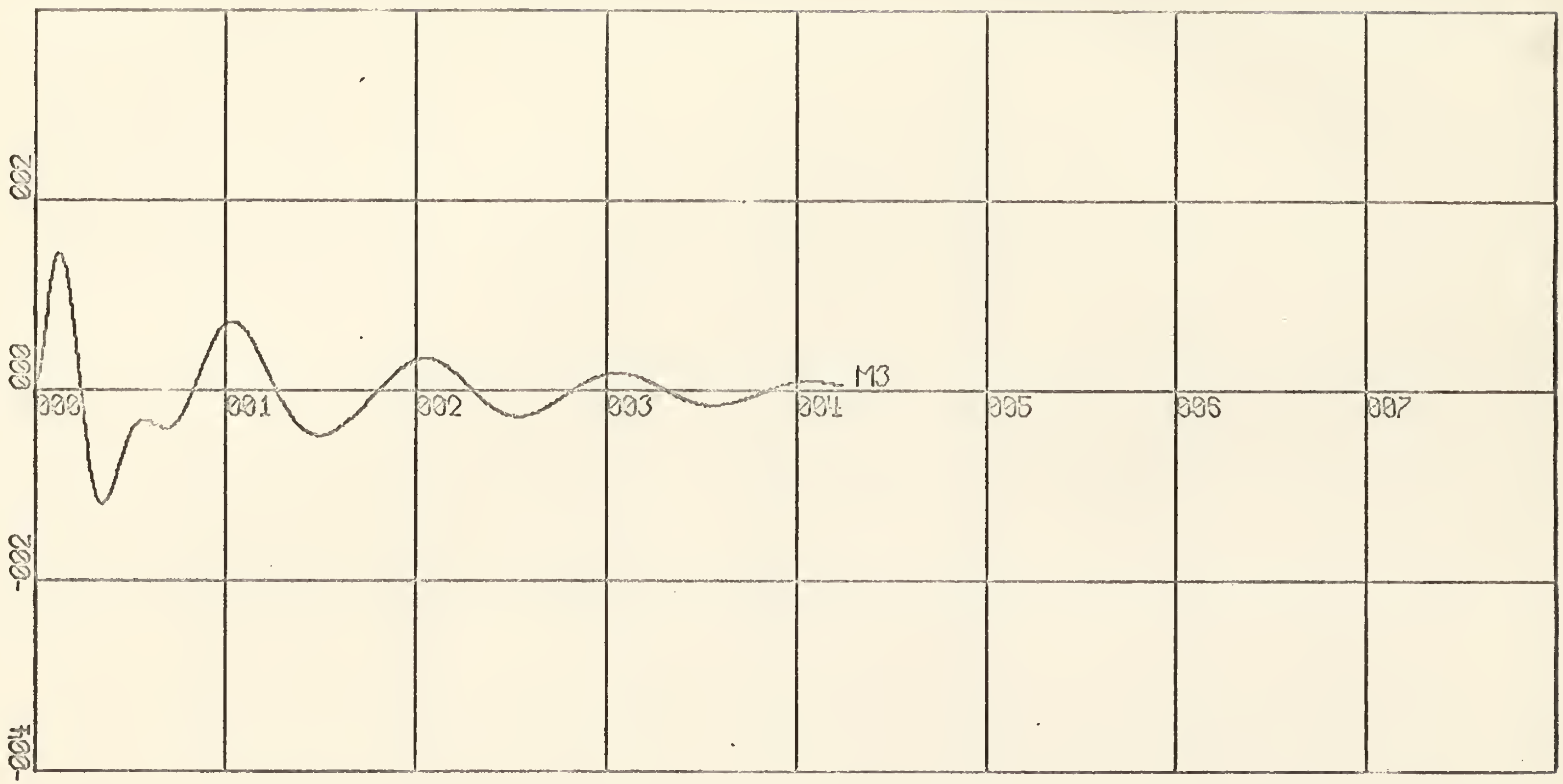
MIKLOS, T.J. JOB BOX M
TRANSIENT PLUS STEADY STATE



X-SCALE = 1.00E+00 UNITS/INCH.
Y-SCALE = 2.00E-01 UNITS/INCH.

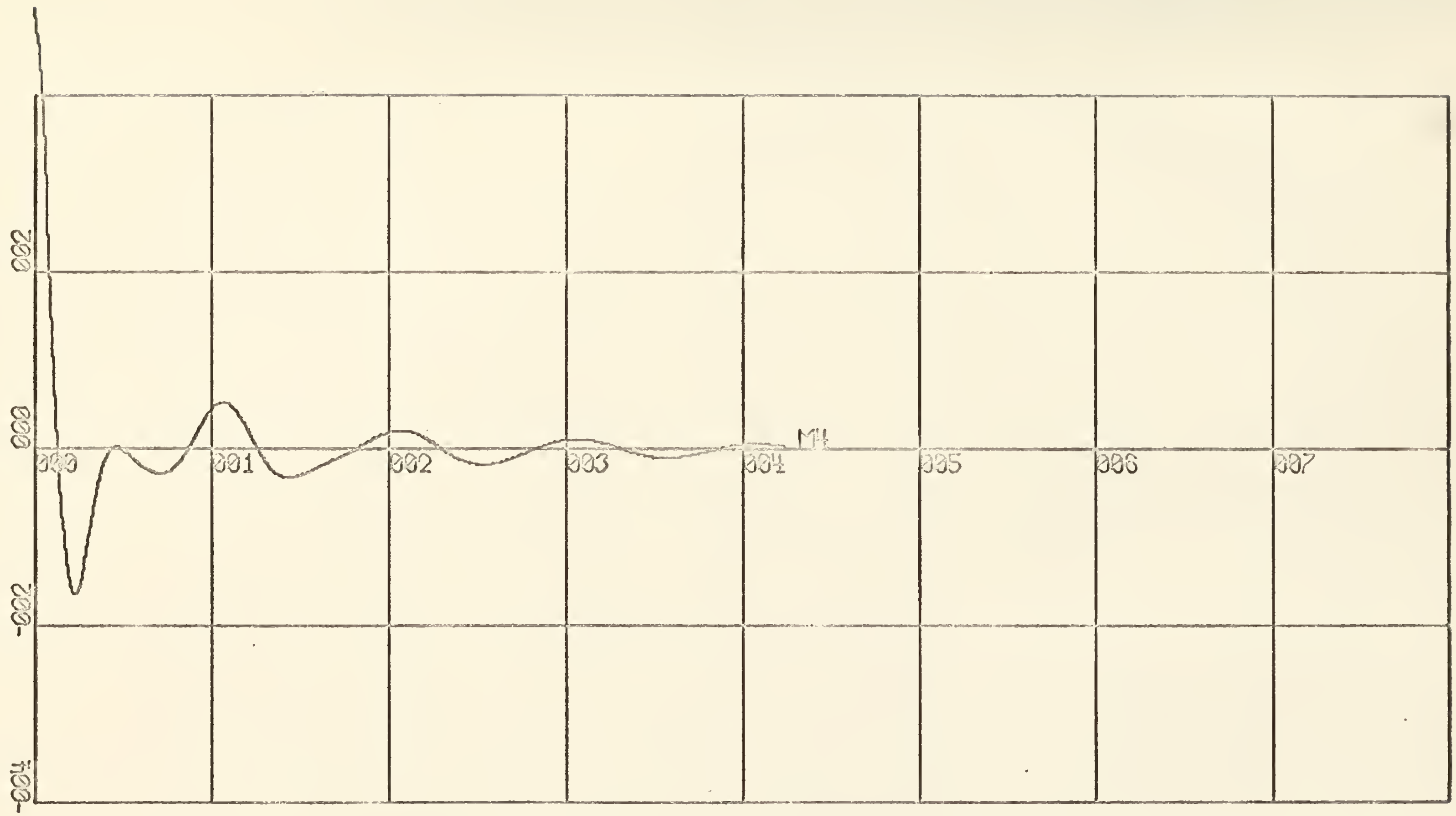
MIKLOS, T.J. JOB BOX M

TRANSIENT PLUS STEADY STATE



X-SCALE = 1.00E+00 UNITS/INCH.
Y-SCALE = 2.00E-01 UNITS/INCH.

MIKLOS, T.J. JOB BOX M
TRANSIENT PLUS STEADY STATE



X-SCALE = 1.00E+00 UNITS/INCH.
Y-SCALE = 2.00E-01 UNITS/INCH.

MIKLOS, T.J. JOB BOX M
TRANSIENT PLUS STEADY STATE